(4) gyökfüggvény

$f(x) = \sqrt[n]{x}$ ($x \geq 0$ ha n páros, $x \in \mathbb{R}$ ha n páratlan)

$\forall a \in \mathbb{R}$

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

$$\lim_{x \to \infty} \sqrt[n]{x} = +\infty$$

n páratlan $\lim_{x \to -\infty} \sqrt[n]{x} = -\infty$

$n=2$ esetén igazoljuk

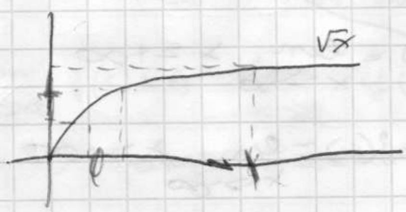
$$\lim_{x \to a} \sqrt{x} = \sqrt{a} \quad (a \geq 0)$$

$\Updownarrow$

$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0$ ha $x \geq 0$, $x \neq a$ és $|x-a| < \delta \Rightarrow |\sqrt{x} - \sqrt{a}| < \varepsilon$

$$|\sqrt{x} - \sqrt{a}| = \frac{(\sqrt{x}+\sqrt{a})\,|\sqrt{x}-\sqrt{a}|}{\sqrt{x}+\sqrt{a}} =$$

$$= \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \underset{\sqrt{x} \geq 0}{\leq} \frac{|x-a|}{\sqrt{a}+0} < \varepsilon$$



$\Updownarrow$

$|x-a| < \sqrt{a} \cdot \varepsilon \qquad \delta = \sqrt{a} \cdot \varepsilon$

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41/2

a) $\lim_{x \to 0} \frac{1}{1+x} = \frac{1}{1} = 1$

~~b)~~ $\cancel{\lim_{x \to 0}}$

b) $\lim_{x \to 0} \frac{x^3 - 3x + 1}{x-4} = -\frac{1}{4}$

c) $\lim_{x \to 0} \frac{x^4 + 3x^2}{x^5 + x^3 + 2x^2} = \lim_{x \to 0} \frac{x^2(x^2+3)}{x^2(x^3+x+2)} = \frac{3}{2}$

d) $\lim_{x \to 1} \frac{x^4 + 2x^2 - 3}{x^2 - 3x + 2} = \text{"}\frac{0}{0}\text{"} \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)(x^2+3)}{\cancel{(x-1)}(x-2)} = \lim_{x \to 1} \frac{(x+1)(x^2+3)}{(x-2)} = -8$

1 mód

$x^4 + 2x^2 - 3 = x^4 - 1 + 2x^2 - 2 = (x^2-1)(x^2+1) + 2 \cdot (x^2-1) = (x^2-1) \cdot (x^2+1+2) =$

$= (x-1) \cdot (x+1) \cdot (x^2+3)$

2, egyező

$(x^2)^2+2x^2-3=0$

$(x^2)_{1,2}=\frac{-2\pm\sqrt{4+12}}{2}=\frac{-2\pm 4}{2}$ $\nearrow -3$, $\searrow 1$ $\Rightarrow x^4+2x^2-3=(x^2-1)\cdot(x^2+3)$

(d) $\lim\limits_{x\to\infty}\frac{x^2-1}{2x^2+1}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{x\to\infty}\frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}}=\frac{1}{2}$ $\quad \frac{1}{x^2}\to 0$

41/3

$\lim\limits_{x\to 1-}\frac{x-1}{|x-1|}=\lim\limits_{x\to 1-}\frac{x-1}{-(x-1)}=\lim\limits_{x\to 1-}\frac{1}{-1}=\underline{-1}$

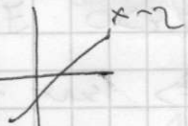
$\Updownarrow$

$x\to 1$

$x<1\Rightarrow x-1<0$

$|x-1|=-(x-1)$

b, $\lim\limits_{x\to 2-}\frac{x}{x-2}\ \text{"}\frac{2}{0}\text{"}\ -\infty$



$\Updownarrow$

$x\to 2$

$x<2$

$\lim\limits_{x\to 0}\frac{x+5}{x^2-3x^3}\overset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\frac{x(x+5)}{x^2(1-3x)}=\lim\limits_{x\to 0}\frac{x+5}{x\cdot(1-3x)}=$ Nem létezik h.e.

$\lim\limits_{x\to 0-}\frac{x+5}{x(1-3x)}=\underline{-\infty}$ $\quad (x+5\to 5,\ x\to 0,\ 1-3x\to 1,\ \frac{5}{0})$

$x<0$

$\lim\limits_{x\to 0+}\frac{x+5}{x(1-3x)}=\underline{+\infty}$ $\quad (x+5\to 5,\ x\to 0^+,\ 1-3x\to 1,\ \frac{5}{0^+})$

42/5 a, $\lim\limits_{x\to -\infty}\frac{2x+1}{\sqrt{x^2+1}}\overset{\frac{-\infty}{\infty}}{=}\lim\limits_{x\to -\infty}\frac{-2-\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}}=\underline{-2}$ $\quad \frac{1}{x^2}\to 0$

$x\to -\infty\Rightarrow$ tfh. $x<0$

$\sqrt{x^2}=|x|=-x$

b, $\lim\limits_{x\to\infty}\frac{x^2+x+1}{\sqrt{x^4+1}}=$

c, $\lim\limits_{x\to\infty}\sqrt{x^2+x}-x=\lim\limits_{x\to\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}=$ $\rightarrow$

$$\to \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}$$

g) $\lim\limits_{x\to+\infty} x\cdot(\sqrt{1+x^2}-x)$

Trigonometrikus függvények határértéke

1) Sinus cosinus függv.

Tétel $\forall a\in\mathbb{R}$ $\lim\limits_{x\to a} \sin x = \sin a$

$\lim\limits_{x\to a} \cos x = \cos a$

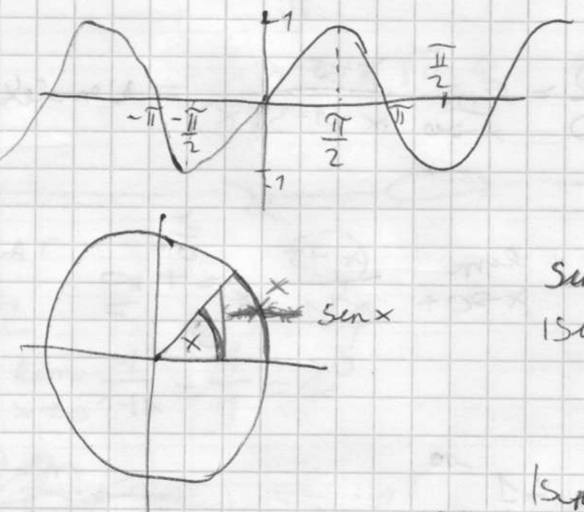
Biz.

1. lépés $a=0$-ra telj. az állítás

$\lim \sin x = \sin 0 = 0 \overset{def}{\iff} \forall \varepsilon>0 \;\exists \delta=\delta(\varepsilon)>0$ ha $x\in\mathbb{R}$ $x\neq 0,\infty$

$|x-0|<\delta \Rightarrow |\sin x - \sin 0|<\varepsilon$

$|x|<\delta$ $\quad |\sin x|<\varepsilon$



$\sin x < x$

$|\sin x|<|x|$ ha

$x\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$|\sin x|<|x|<\varepsilon$

$|x|<\frac{\pi}{2}$

$\delta := \min\left\{\frac{\pi}{2}, \varepsilon\right\}$ // sin x

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$\lim\limits_{x\to 0} \cos x = \cos 0 = 1 \overset{def}{\iff} \forall\varepsilon>0 \;\exists\delta=\delta(\varepsilon)>0$

ha $x\neq 0$ $\quad |x|<\delta \Rightarrow |\cos x - 1|<\varepsilon$

$$|\cos x - 1| = \frac{|\cos x - 1|\cdot|\cos x+1|}{|\cos x+1|} = \frac{|\cos^2 x - 1^2|}{|\cos x+1|} = \to$$

$x<\frac{\pi}{2}$ $\qquad (a-b)(a+b)=a^2-b^2$

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$\frac{\sin x}{x}$ -re visszavezető feladatok:

42/6 a/i/l/m számmal.

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$\lim_{x\to a}\frac{\sin(x)}{n x}=1 \qquad \lim_{x\to a} n(x)=0$$

(a) $\lim_{x\to 0}\frac{\sin 3x}{x}\cdot 3=3$

$$\lim_{x\to 0}\frac{\sin 7x}{7x}\cdot 7=7$$

(c) $\lim_{x\to 0}\frac{\sin 3x}{\sin 6x}=\lim_{x\to 0}\frac{\frac{\sin 3x}{3x}\cdot 3x}{\frac{\sin 6x}{6x}\cdot 6x}=\frac{3}{6}=\frac{1}{2}$

$a, b\neq 0$

$$\lim\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\frac{\frac{\sin ax}{ax}\cdot ax}{\frac{\sin bx}{bx}\cdot bx}=\frac{a}{b}$$

$$\lim_{x\to 0}\frac{\sin 2x}{\operatorname{tg} 3x}=\lim_{x\to 0}\frac{\sin 2x}{\frac{\sin 3x}{\cos 3x}}=\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}\cdot\cos 3x=$$

$$=\lim_{x\to 0}\frac{\frac{\sin 2x}{2x}\cdot 2x}{\frac{\sin 3x}{3x}\cdot 3x}\cdot\cos 3x=\frac{2}{3}$$

$\boxed{\text{Hf } \lim_{x\to 0}\frac{\operatorname{tg} 5x}{\operatorname{tg} 7x}}$

(l) $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}=\lim_{x\to 0}$

$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{1}{1+\cos x}=\frac{1}{2}$$

(m) $\lim_{x\to 0}\frac{\sqrt{\cos x}-1}{x^2}=\lim_{x\to 0}\frac{\cos x-1}{x^2(\sqrt{\cos x}+1)}=\lim_{x\to 0}\frac{\cos^2 x-1}{x^2(\sqrt{\cos x}+1)(\cos x+1)}=$

$$=\lim_{x\to 0}\frac{-\sin^2 x}{x^2(\sqrt{\cos x}+1)(\cos x+1)}=\lim_{x\to 0}-\left(\frac{\sin x}{x}\right)^2\frac{1}{\sqrt{\cos x}+1}\cdot\frac{1}{\cos x+1}=-\frac{1}{4}$$

(h) $\lim_{x \to 0} \frac{1}{\operatorname{sen} x} - \frac{1}{\operatorname{tg} x} = \lim_{x \to 0} \frac{1}{\operatorname{sen} x} - \frac{\cos x}{\operatorname{sen} x} = \lim_{x \to 0} \frac{(1-\cos x)(1+\cos x)}{\operatorname{sen} x(1+\cos x)} = \cancel{\lim_{x \to 0}}$

$= \lim_{x \to 0} \frac{1-\cos x^2}{\operatorname{sen} x(1+\cos x)} = \cancel{\lim_{x \to 0}} \frac{\operatorname{sen} x^2}{\cancel{\operatorname{sen} x^2}} = \lim_{x \to 0} \frac{\operatorname{sen}^2 x}{\operatorname{sen} x(1+\cos x)} = \lim_{x \to 0} \frac{\operatorname{sen} x}{1+\cos x}$

$\operatorname{sen} x \frac{1}{1+\cos x} = 0$

Megjegyzés $f_1 \dots f_n \in D_a \Rightarrow f_1 + \dots + f_n \in D_a$

$f_1 \cdot \dots \cdot f_n \in D_a$

Tétel $g: M \to K$ $f: K \to \mathbb{R}$ $(M\ K\ \mathbb{R})$

$a \in M$ esetén $g \in D_a$ és $f \in D_{g(a)} \Rightarrow f \circ g \in D_a$

és $(f(g(x)))' \Big|_{x=a} = f'(g(a))\, g'(a)$

láncszabály

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$(f+g)' = f' + g'$

$(\lambda \cdot g)' = \lambda g'$

$(f \cdot g)' = f' \cdot g + f \cdot g'$

$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f g'}{g^2}$

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

1) $f(x) = 3x^5 + 4x^3 + 6x^2 + 7x + 2$

$f'(x) = 3 \cdot 5 \cdot x^4 + 4 \cdot 3 \cdot x^2 + 6 \cdot 2x + 7 \cdot 1 x^0 + 0 =$
$= 15x^4 + 12x^2 + 12x + 7$

2, $f(x) = \frac{3x^2 + 6x + 2}{2x - 1}$

$f'(x) = \frac{(3 \cdot 2x + 6 \cdot x^0 + 0) \cdot (2x-1) - (3x^2 + 6x + 2)\, 2}{(2x-1)^2}$

$(\sin x)' = \cos x$

3, $f(x) = \operatorname{tg} x \left(= \frac{\sin x}{\cos x}\right)$

$(\cos x)' = -\sin x$

$f'(x) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

HF

$\ln' x = \frac{1}{x}$

$x' = 1$

4, $f(x) = \operatorname{ctg} x \left(= \frac{\cos x}{\sin x}\right)$

$f'(x) =$

5, $f(x) = x \ln x$

$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$

6, $f(x) = \log_a x \left(= \frac{\ln x}{\ln a}\right)$

$f(x) = \frac{1}{\ln a} \cdot \ln x$

$f'(x) = \frac{1}{\ln a} \cdot (\ln x)'$

7, $f(x) = \sin(7x^2)$

$f(x) = \cos(7x^2) \cdot 7 \cdot 2x$

8, $f(x) = \cos^2 x = (\cos x)^2$

$f(x) = 2 \cdot \cos x \cdot (-\sin x)$

9, $f(x) = \ln(\sin x)$

$f'(x) = \frac{1}{\sin x} \cdot \cos x$

(10) $f(x) = (6x^2 - 3x + 1)^3$

$f'(x) = 3(6x^2 - 3x + 1)^2 \cdot (12x - 3)$

$(6 \cdot 2 \cdot x - 3 \cdot 1 + 0)$ //

Inverz függvény differenciálási szabálya

Tétel

$f: (\alpha, \beta) \to (a, b)$ f szigorúan monoton

$\gamma \in (\alpha, \beta)$ $f \in D_\gamma$ $f(\gamma) \neq 0$

$\Rightarrow f(\gamma) = c$ $f^{-1} = D_c$ és

$$(f^{-1})' = \frac{1}{f'(f^{-1}(c))} = \frac{1}{f(\gamma)}$$

Biz //

Példák (1) $f(x) = e^x$ $(x \in \mathbb{R})$ $f \in D_\mathbb{R}$

$f(x)$?

$f^{-1}(x) = \ln x$ $(x > 0)$

$\Leftrightarrow f(x)\ \ln x$ $(x > 0)$

$f^{-1}(x) = e^x$ $(x \in \mathbb{R})$

$f \in D_\mathbb{R}$ és $(\ln x)' = \frac{1}{x} = 0$

$\Rightarrow f^{-1}(x) = e^x \in D_\mathbb{R}$

$$(f^{-1})(x) = \frac{1}{f'(f^{-1}(x))}$$

Példa

$h(x) = x^x = e^{\ln x^x} = e^{x \ln x} \quad x>0$

$h'(x) = (e^{x \cdot \ln x})' = e^{x \ln x}(x \cdot \ln x)' = x^x(x' \ln x + x \ln x)') =$

$= x^x(1 \cdot \ln x + x \cdot \frac{1}{x}) = \boxed{x^x(\ln x + 1)}$

Differenciálás

(1) $f(x) = x \cdot \arcsin x$

$f'(x) = (x') \cdot \arcsin x + x(\arcsin x)' = 1 \arcsin x + x \frac{1}{\sqrt{1-x^2}}$

2) $f(x) = \frac{x^2+2x-1}{e^x}$

$f'(x) = \frac{(2x+2 \cdot 1+0)e^x - (x^2+2x-1)(e^x)}{(e^x)^2}$

3 $f(x) = \sqrt{x} \operatorname{arctg} x$

$f'(x) = (x^{\frac{1}{2}}) \operatorname{arctg} x + \sqrt{x} \cdot (\operatorname{arctg} x)' =$

$= \cancel{\frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2}}$

4, $f(x) = \arcsin(8x^2 - 7x + 1)$

$f'(x) = \frac{1}{\sqrt{1-(8x^2-7x+1)}} \cdot (8x^2-7x+1)'$

$\frac{1}{\sqrt{1-(8x^2-7x+1)}}(16x-7)$

5, $f(x) = e^{x \sin x}$

$f'(x)$

6, $f(x) = e^{x^2-1}$

$f'(x) =$

7 $f(x) = x^{\sqrt{x}}$

$f'(x)$

3, $\frac{1}{2} \cdot \frac{1}{\sqrt{x}} \cdot \operatorname{arctg} x + \sqrt{x} \frac{1}{1+x^2}$

$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f g'}{g^2}$

$(f \cdot g)' = f' \cdot g + f \cdot g'$

$(\lambda \cdot g)' = \lambda \cdot f'$

$(f+g)' = f' + g'$

$(f(g(x)))' = f'(g(x)) \, g'(x)$

$(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2} \cdot x^{-\frac{1}{2}} =$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$

Összetett fgv

~~$f(g(x))$~~

$f(g(x))' = f'(g(x)) \, g'(x)$

$\sqrt{x} = (x^{\frac{1}{2}}) = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2} \cdot x^{-\frac{1}{2}} =$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$

$f(x) = e^{x \cdot \operatorname{sen} x}$

$\cancel{f(x)\, e^{x \cdot \operatorname{sen} x} \; (\ldots)' =}$

$f'(x) = e^{x \cdot \operatorname{sen} x} \; (x \cdot \operatorname{sen} x)' = e^{x \operatorname{sen} x} \left(x' \operatorname{sen} x + x (\operatorname{sen} x)' \right) =$

$= e^{x \cdot \operatorname{sen} x} (1 \cdot \operatorname{sen} x + x \cdot \cos x)$

$f(x) = e^{x^2 - 1}$

$f'(x) = e^{x^2 - 1} \; (x^2 - 1)' = e^{x^2 - 1} \cdot 2x$

$f(x) = x^{\sqrt{x}}$

$f(x) = e^{\ln x \sqrt{x}} = e^{\sqrt{x} \ln x}$

$f'(x) = e^{\sqrt{x} \ln x} \cdot (\sqrt{x} \ln x) = x^{\sqrt{x}} \cdot \left(\underbrace{\frac{1}{2} \cdot \frac{1}{\sqrt{}}}_{\sqrt{x}} \cdot \ln x + \sqrt{x} \cdot \underbrace{\frac{1}{x}}_{\ln(x)} \right)$

$f(x) = \operatorname{sen} (e^{x^2 - 1})$

$f'(x) = \cos (e^{x^2 - 1}) \cdot (e^{x^2 - 1})' = \cos (e^{x^2 - 1}) \cdot x^2 - 1 \cdot 2x$

$f(x) = \sqrt{\operatorname{sen} x}$

$\cancel{f(x)' = \operatorname{sen} / \tfrac{1}{2}\, x^{\frac{1}{2}} = \cos \tfrac{1}{2} x^{\frac{1}{2} - 1} = \cos \tfrac{1}{2} \cdot x^{-\frac{1}{2}} = \cos \tfrac{1}{2} \cdot \tfrac{1}{\sqrt{x}}}$

$f(x)' = \frac{1}{2} \cdot \frac{1}{\sqrt{\operatorname{sen} x}} \cdot \cos x$

$f(x) = (6x^2 - 2x + 5)^4$

$f'(x) = 4 (6x^2 - 2x + 5)^3 \cdot (6x^2 - 2x + 5)' = 4 \cdot (6x^2 - 2x + 5)^3 \cdot 12x - 2$

Határozza meg az alábbi függvények grafikonjának x_0 pontbeli húzott érintő egyenesének egyenletét. Mekkora szöget zár be az érintő az x tengely pozitív felével

$x_0 = \frac{\pi}{4}$ $\quad y_0 = f(x_0) = \cos x_0 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

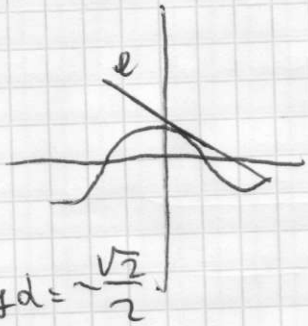
a, $f(x) = \cos x$

$f'(x) = -\sin\frac{\pi}{4} = \boxed{-\frac{\sqrt{2}}{2} = m}$

$e: y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\cdot(x - \frac{\pi}{4})$

(b, ~~$f(x) = \sin(3x^2 + \frac{\pi}{2})$ $x_0 = 0$~~)

$$\boxed{y = -\frac{\sqrt{2}}{2}\cdot(x - \frac{\pi}{4}) + \frac{\sqrt{2}}{2}}$$



$tg\,\alpha = -\frac{\sqrt{2}}{2}$

$e: y - y_0 = f'(x_0)\cdot(x - x_0)$

$y_0 = f(x_0)$

$m = f'(x_0) = tg\,\alpha$

$\alpha = arc\,tg\left(-\frac{\sqrt{2}}{2}\right) \approx -35{,}26^\circ$

b, $f(x) = \sin(3x^2 + \frac{\pi}{2})$ $\quad x_0 = 0$

$f'(x) = \cos(3x^2 + \frac{\pi}{2})\cdot(3x^2 + \frac{\pi}{2})' = \cos(3x^2 + \frac{\pi}{2})\cdot(6x)$

$m = tg\,\alpha = f'(x_0) = f'(0) = 0$ $\quad y_0 = f(x_0) = f\ \sin\frac{\pi}{2} = 1$ $\quad \alpha = 0^\circ$

~~(c) $f(x) = x^{\frac{x}{2}}$ $x_0 = 1$~~

$e: y - 1 = 0\cdot(x - 0)$

$y - 1 = 0$

$y = 1$

(c) $f(x) = x^{\frac{x}{2}} = e^{\ln x^{\frac{x}{2}}}$ ~~xxx~~ $e^{\frac{x}{2}\ln x}\cdot(\frac{x}{2}\ln x)'$

$f'(x) = e^{\frac{x}{2}\ln x}\,\frac{1}{2}\cdot(\cancel{x}\frac{}{2}\cdot\frac{1}{x}) = e^{\frac{x}{2}\ln x}\cdot\frac{1}{2}$

$f'(x)\ e^{\frac{x}{2}\ln x}\left(\frac{1}{2}\cdot\ln x(\frac{x}{2}\cdot\frac{1}{x}\right) = e^{\frac{x}{2}\ln x}\left(\frac{1}{2}\ln + \frac{1}{2}\right)$

$= e^{\frac{x}{2}\cdot\ln x}\left(\frac{1}{2}\cancel{\ln x} + \frac{x}{2}\cdot\frac{1}{x}\right. = x^{\frac{x}{2}}\frac{1}{2}(\ln x + 1$

$f'(x)\ e^{\frac{x}{2}\ln x}\ (\frac{x}{2}\ln x)' $

$m = f'(x_0)\ f'(1) = 1^{\frac{1}{2}}\cdot\frac{1}{2}\ (\ln 1 + 1 = \frac{1}{2}$

$f(x_0)\ f(1) = 1^{\frac{1}{2}} = 1$

$y - 1 = \frac{1}{2}\cdot x - 1$ $\quad y = \frac{1}{2}x + \frac{1}{2}$ $\quad tg\,\alpha = m$ $\quad \alpha = arc\,tg\,\frac{1}{2} \approx 26{,}56^\circ$