0

FELADOTOK óra

(0) $(a_n = c, n \in \mathbb{N}) \Rightarrow (a_n, n \in \mathbb{N})$ konvergens és határértéke c

Biz

(B) $\forall \varepsilon > 0 \ \exists N = N(\varepsilon) \in \mathbb{N}$ ha

$n > N$ akkor $|a_n - c| < \varepsilon$

$a_n = c \ (n \in \mathbb{N})$

$|c - c| = 0 < \varepsilon \Rightarrow \forall N \in \mathbb{N}$ jó

2) $(a_n = \frac{1}{n}, n \in \mathbb{N}^*)$ harmonikus

Ezért $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{1}{n} = 0$

Biz $\forall \varepsilon > 0 \ \exists N = N(\varepsilon) \in \mathbb{N}$; $n > N \Rightarrow |\frac{1}{n} - 0| < \varepsilon$

$|\frac{1}{n} - 0| < \varepsilon$

$|\frac{1}{n}| < \varepsilon \quad n \in \mathbb{N}^*$

$\frac{1}{n} < \varepsilon$

egészrész $n > \frac{1}{\varepsilon}$

$N = \left[\frac{1}{\varepsilon}\right] + 1$

$n > \left[\frac{1}{\varepsilon}\right] + 1 = N \Rightarrow n > N > \frac{1}{\varepsilon} \Rightarrow |\frac{1}{n} - 0| < \varepsilon$

Vegyes Feladatatar!

(1) Vizsgáljuk meg az alábbi sorozatok monotonitás szempontjából

(a) $(a_n = n! \ ; n \in \mathbb{N})$

$\boxed{n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}$

$\boxed{0! = 1}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}(n+1)}{\cancel{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}} =$

$= n + 1 \geq 1 \ (\forall n \in \mathbb{N})$

$a_{n+1} \geq a_n \ (\forall n \in \mathbb{N})$

$(a_n \ n \in \mathbb{N})$ monoton nő

$(a-b)(a+b) = a^2 - b^2$

b, $(a_n = \sqrt{n}, n \in \mathbb{N})$

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} =$$
$$= \frac{(\sqrt{n+1}-\sqrt{n})\cdot(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\underbrace{\underset{\vee\atop 0}{\sqrt{n+1}} + \underset{\vee\atop 0}{\sqrt{n}}}_{\vee\atop 0}} \xrightarrow{\;n\;}$$

$> 0 \; (\forall n \in \mathbb{N})$

$\Rightarrow a_{n+1} - a_n > 0 \; (\forall n \in \mathbb{N})$

$a_{n+1} > a_n \; (\forall n \in \mathbb{N})$

$a_n (n \in \mathbb{N})$ szig mon nő!

2, Definíció alapján határozzuk meg az alábbi sorozat határértékét

$\left(a_n = \frac{2n+3}{3n-5}, n \in \mathbb{N}\right)$ sejtés $\lim\limits_{n\to\infty} a_n = \frac{2}{3}$

Bz $\forall \varepsilon > 0 \; \exists N = N(\varepsilon) \in \mathbb{N}$, ha $n > N \Rightarrow$

$$\left|\frac{2n+3}{3n-5} - \frac{2}{3}\right| < \varepsilon$$

RACSKÓ DÓRI

$$\left|\frac{3(2n+3) - 2(3n-5)}{3\cdot(3n-5)}\right| = \left|\frac{6n+9-6n+10}{3\cdot(3n-5)}\right| = \left|\frac{19}{3\cdot(3n-5)}\right|$$

$$\left|\frac{19}{3\cdot(3n-5)}\right| < \varepsilon \qquad \boxed{n > 1}$$

$\vee$
0

$$\left|\frac{19}{3(3n-5)}\right| < \varepsilon \qquad |\cdot 3n-5$$

$$\left|\frac{19}{3}\right| < \varepsilon \cdot (3n-5) \qquad |: \varepsilon \; (>0)$$

$$\left|\frac{19}{3\varepsilon}\right| < 3n-5 \qquad |+5$$

$$\left|\frac{19}{3\cdot\varepsilon} + 5\right| < 3n \qquad |:3$$

$$\left(\frac{19}{3\varepsilon} + 5\right)\frac{1}{3} < n$$

$$\left|\left(\frac{19}{3\varepsilon}+5\right)\cdot\frac{1}{3}\right| < n$$

$N :=$
$\max\left\{1, \left[\left(\frac{19}{3\varepsilon}+5\right)\frac{1}{3}\right]+1\right\}$

$= \frac{1}{b_n} \cdot |a_n - a| + \frac{|a|}{|b_n||b|} \cdot |b_n - b|$

tudjuk hogy

$\exists M > 0$

$|b_n| < M \ \forall n \in \mathbb{N}$ viszont mivel a $b \neq 0$ $\exists m > 0$ és $\bar{N} \in \mathbb{N}$

$|b_n| > m > 0$ ha $n > \bar{N}$

$\Rightarrow n > \bar{N} \ \frac{1}{|b_n|} < \frac{1}{m}$

$\leq \frac{1}{m} \cdot |a_n - a| + \frac{|a|}{m|b|} |b_n - b|$

$(o)\ \varepsilon_a = \frac{m \cdot \varepsilon}{2}$

(oo)-ban feltéve hogy $a \neq 0$

$\varepsilon_a = \frac{m \cdot |b| \cdot \varepsilon}{2 \cdot |a|}$

GYAKORLAT

1, $a_n = \frac{5n+6}{3n-2}$ $(n \in \mathbb{N})$ Monotonitás korlátosság konvergencia

Monotonitás

$a_{n+1} - a_n = \frac{5(n+1)+6}{3(n+1)-2} - \frac{5n+6}{3n-2} = \frac{5n+5+6}{3n+3-2} - \frac{5n+6}{3n-2} = \frac{(3n-2)(5n+5+6)}{(3n-2)\cdot(3n+3-2)}$

$- \frac{(3n+3-2)\cdot(5n+6)}{(3n+3-2)(3n-2)} \qquad \frac{(5n+11)\cdot(3n-2) - (5n+6)\cdot(3n+1)}{(3n+1)(3n-2)} =$

$= \frac{15n^2+33n-10n-22}{(3n+1)(3n-2)} = \frac{-28}{(3n+1)(3n-2)} < 0$ ha $n \geq 1 \Rightarrow a_n$ szig. mon. csökk. ha $n \geq 1$

$(3n+1) > 0$, $(3n-2) > 0$

Konvergencia

Sejtés $\lim_{n\to\infty} a_n = \frac{5}{3}$

Biz $\forall \varepsilon > 0 \ \exists N = N(\varepsilon) \in \mathbb{N}$

ha $n > N$ akkor

$\left| \frac{5n+6}{3n-2} - \frac{5}{3} \right| < \varepsilon$

$\left| \frac{3(5n+6) - (3n-2)5}{3(3n-2)} \right| < \varepsilon$

$\left| \frac{15n+18-15n+10}{9n-6} \right| < \varepsilon$

$\left| \frac{28}{9n-6} \right| < \varepsilon$ tfh $n \geq 1$

$\frac{28}{9n-6} < \varepsilon$

$\frac{28}{3 \cdot \varepsilon} < \varepsilon n - 2$

$\frac{\frac{28}{3\varepsilon} + 2}{3} < n$

$N = \left[\frac{\frac{28}{3\varepsilon} + 2}{3} \right] + 1$

Korlátosság

Mivel $(a_n \; n \in \mathbb{N})$ konvergens ezért korlátos

Mivel —ll— szig mon ~~csökk~~ növ ha $n \geq 1$ ezért

$\sup\{a_n \; n \in \mathbb{N}\} = \max\{a_0, a_1\} = \max\{-3, 11\} = \underline{11}$

Mivel $(a_n \; n \in \mathbb{N})$ szig mon nő ha $n \geq 1$ és korlátos ezért

$\inf\{a_n, n \geq 1\} = \lim_{n \to \infty} a_n = \frac{5}{3}$

$\Rightarrow \inf\{a_n \; n \in \mathbb{N}\} = \min\{a_0 \frac{5}{3}\} = \min\{-3 \frac{5}{3}\} = \underline{-3}$

(2) $\left(a_n = \frac{-2n+3}{3n+1}, n \in \mathbb{N}\right)$

Monoton.

$a_{n+1} - a_n = \frac{-2(n+1)+3}{3(n+1)+1} - \frac{-2n+3}{3n+1} =$

$\frac{-2n+1}{3(n+1)-1} - \frac{-2n+3}{3n+1} = \frac{(3n+1)\cdot(2n+2) - (3(n+1)-1)\cdot(-2n+3)}{(3n+1)\cdot[3(n+1)-1]}$

$\frac{6n^2-2n+3n+1-(6n^2-8n+9n+12)}{3n+1} = \frac{-11}{\underset{\vee_0}{(3n+1)}\underset{\vee_0}{(3n+1)}} < 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow a_n$ szig mon csökk $\forall n \in \mathbb{N}$

Konvergencia Sejtés $\lim_{n \to \infty} a_n = -\frac{2}{3}$

$\varepsilon > 0 \; \exists N = N(\varepsilon) \in \mathbb{N}$

ha $n > N$, akkor

$\left|\frac{2n+3}{3n+1} - \left(-\frac{2}{3}\right)\right| < \varepsilon$

$\left|\frac{-2n+3}{3n+1} + \frac{2}{3}\right| < \varepsilon$

$\left|\frac{3(-2n+3)+2(3n+1)}{(3n+1)3}\right| < \varepsilon$

$\frac{-6n+9+6n+2}{(3n+1)3}$

$\frac{11}{(3n+1)3} < \varepsilon$

$\frac{11}{3\varepsilon} < 3n+1$

$N = \left[\frac{\frac{11}{3\varepsilon} - 1}{3}\right] + 1$

GYAK.

Sorozatok határértéke nézeti tulajdonság alapján

1, $\frac{3n^2-2n+6}{-4n^2+6n+5} \overset{\frac{\infty}{\infty}}{} = \frac{3-\frac{2}{n}+\frac{6}{n^2}}{-4+\frac{6}{n}+\frac{5}{n^2}} \to \boxed{\frac{-3}{4}}$ $(n\to\infty)$

$\frac{3n^2-2^n+6}{n-4} \overset{\frac{\infty}{\infty}}{} = \frac{3n-2^2+\frac{6}{n}}{1-\frac{4}{n}} \to \boxed{\infty} (n\to\infty)$

$\frac{3n+7}{n^2-6n+2} \overset{\frac{\infty}{\infty}}{} = \frac{\frac{3}{n}+\frac{7}{n^2}}{1-\frac{6}{n}+\frac{2}{n^2}} \to \boxed{0}$ $n\to\infty$

legmagasabb kitevőjű tagnak a nevezőben kell elosztani a számlálót és a nevezőt is

2, $\frac{n^2+1}{2n+1} - \frac{3n^2+1}{6n+1} = \cancel{(2+1)\cdot(6n+1)-(2n+1)\cdot(6n+1)}$

$6n^3+n^2+6n+1-(6n^3+2n+3n^2+1)$

$\frac{(n^2+1)\cdot(6n+1)-(3n^2+1)\cdot(2n+1)}{(2n+1)\cdot(6n+1)} = \frac{2n^2+4n}{(2n+1)\cdot(6n+1)} = \frac{-2+\frac{4}{n}}{\frac{n}{n^2}\cdot(2+\frac{1}{n})\cdot(6+\frac{1}{n})} \to \frac{-2}{12} = \frac{1}{6}$

$\frac{(2n+1)(6n+1)}{n^2} = \frac{(2n+1)(6n+1)}{n\cdot n}$

3) $\frac{(n+4)^3-n\cdot(n+6)^2}{n^3} =$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$(a+b)^2 = a^2+2ab+b^2$

$\cancel{(n^3+3\cdot n^2\cdot 4)} = \frac{n^3+3n^2\cdot 4+3\cdot n\cdot 16+4^3-n\cdot(n^2+12n+36)}{n^3} =$

$= \frac{12n^2+48n+64-12n^2-36n}{n^3} = \frac{12n+64}{n^3} = \frac{\frac{12}{n^2}+\frac{64}{n^3}}{\frac{n^3}{n^3}} \to \frac{0}{1} = 0$

II. típus Változó tagszámú sorozat.

$\frac{1+2+\dots+n}{n+2} - \frac{n}{2} =$

$1+2+3+\dots+n = \frac{(n+1)n}{2}$

$= \frac{\frac{n\cdot(n+1)}{2}}{n+2} - \frac{n}{2} = \frac{n\cdot(n+1)}{2(n+2)} - \frac{n}{2} =$

$1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$= \frac{n(n+1)-n(n+2)}{2\cdot(n+2)} = \frac{n^2+n-n^2-2n}{2\cdot(n+2)} = \frac{-n}{2(n+2)} = \frac{-1}{2(1+\frac{2}{n})} \to -\frac{1}{2}$ $n\to\infty$

$\frac{1}{3}$

2) $\dfrac{1^2+2^2+3^2+\ldots+n^2}{n^3}$

Mf.

3 $\dfrac{1\cdot 2+2\cdot 3+\ldots+\overbrace{n\cdot(n+1)}^{n^2+n}}{n^3} = \dfrac{(1^2+1)+(2^2+2)+(3^2+3)+\ldots+(n^2+n)}{n^3} =$

$\dfrac{(1^2+2^2+\ldots+n^2)+(1+2+\ldots+n)}{n^3} = \dfrac{\dfrac{n\cdot(n+1)\cdot(2n+1)}{6}+\dfrac{n\cdot(n+1)}{2}}{n^3} =$

$= \dfrac{n\cdot(n+1)\cdot(2n+1)+3n(n+1)}{6n^3} = \dfrac{n\cdot(n+1)\cdot(2n+1+3)}{6n^3} =$

$= \dfrac{n\cdot(n+1)\cdot 2\cdot(n+2)}{3\cdot 2n^3} = \dfrac{n\cdot(n+1)\cdot(n+2)}{3n^3} \cdot \dfrac{\frac{1}{n^3}}{\frac{1}{n^3}}$

$= \dfrac{1\cdot\left(1+\frac{1}{n}\right)\cdot\left(1+\frac{2}{n}\right)}{3} \to \dfrac{1}{3} \quad n\to\infty$

3. típus „q^n"

legnagyobb alapú q^n hatvánnyal egyszerűsítünk

$\dfrac{2^n+3^{-n}+2^{n-1}+3^{n+1}}{2^{-n}-3^n+2^{2n+1}} = \dfrac{2^n+\left(\frac{1}{3}\right)^n+\frac{1}{2}\cdot 2^n+3^n\cdot 3}{\left(\frac{1}{2}\right)^n-3^n+2\cdot 4^n}\cdot\dfrac{\frac{1}{4^n}}{\frac{1}{4^n}}$

$= \dfrac{\left(\frac{2}{4}\right)^n+\left(\frac{1}{12}\right)^n+\frac{1}{2}\cdot\left(\frac{1}{2}\right)^n+3\cdot\left(\frac{3}{4}\right)^n}{\left(\frac{1}{8}\right)^n-\left(\frac{3}{4}\right)^n+2} \to \dfrac{0}{2}=0 \quad (n\to\infty$

$q^n\to 0$ ha $|q|<1$

(7.) b $\sqrt[n]{2\cdot 3^{n+1}-2^{2n-1}\cdot 4+3^{2n}} = \sqrt[n]{2\cdot 3^n\cdot 3-(2^2)^n\cdot(2^{-1})\cdot 4+3^{2n}}$

75
76 $= \sqrt[n]{2\cdot 3^n\cdot 3-(2^2)^n\cdot\frac{1}{2}\cdot 4+(3^2)^n} = \sqrt[n]{6\cdot 3^n-2\cdot 4^n+9^n} =$
78
77 $= \sqrt[n]{9^n}\cdot\sqrt[n]{6\cdot\left(\frac{3}{9}\right)^n-2\cdot\left(\frac{4}{9}\right)^n+1} = 9\cdot\sqrt[n]{6\cdot\left(\frac{1}{3}\right)^n-2\left(\frac{4}{9}\right)^n+1} =$
79
80 rendőrelv

$9\cdot\sqrt[n]{0-2\frac{4}{9}+1} \leq 9\cdot\sqrt[n]{\frac{8}{9}+1} \leq 9\cdot\sqrt[n]{6\left(\frac{1}{3}\right)^n-2\left(\frac{4}{9}\right)^n+1} \leq 9\cdot\sqrt[n]{6+0+1} = 9\sqrt[n]{7}$

$9\cdot\sqrt[n]{\frac{1}{9}} \to \frac{1}{9} \to 9 \qquad \to 9 \qquad \to \frac{1}{9}$

8. tine $\left(1+\frac{1}{n}\right)^n$-re visszavezető feladatok

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ jött $\lim_{n\to\infty}\left(1+\frac{1}{v(n)}\right) = e$ ha $\lim_{n\to\infty} v(n) = \pm\infty$

példák

(1) $\left(1-\frac{1}{n}\right)^n = \underbrace{\left(\left(1+\frac{1}{(-n)}\right)^{(-n)}\right)}_{\to e}^{-1} \to e^{-1} = \frac{1}{e} \quad (n\to\infty)$, $(-n)\to -\infty$

123
124
125
126
129
130
131

144 $\left(\frac{n^2+1}{n^2-2}\right)^{n^2}$

133 $\left(1+\frac{1}{2^n}\right)^{n!}$

(2) $\left(1+\frac{1}{n}\right)^{2n} = \left(\left(1+\frac{1}{n}\right)^n\right)^2 \to e^2 \quad (n\to\infty)$

(3) $\left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} \quad (n\to\infty)$

(4) $\left(1+\frac{1}{2n}\right)^n = \underbrace{\left(\left(1+\frac{1}{2n}\right)^{2n}\right)}_{\to e}^{\frac{1}{2}} \to e^{\frac{1}{2}} = \sqrt{e} \quad (n\to\infty)$, $2n\to\infty$

(5) $\left(1+\frac{1}{n}\right)^{-n} = \left(\left(1+\frac{1}{n}\right)^n\right)^{-1} \to e^{-1}$

6, $\left(1+\frac{1}{n^2}\right)^n = \underbrace{\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)}_{\to e}^{\frac{1}{n}} \to e^0$

ha $a_n \to a > 0$
$b_n \to b \in \mathbb{R}$ $\Rightarrow a_n^{b_n} \to a^b$

(7) $\left(1+\frac{1}{n}\right)^{n^2} = \left(\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{\to e}\right)^{n}$

Tudjuk, hogy

$2 \leq \left(1+\frac{1}{n}\right)^{n} \leq 3$

$2^n \leq \left(\left(1+\frac{1}{n}\right)^{n}\right)^{n} \leq 3^n$

$\downarrow \quad \downarrow \quad \downarrow$

$\infty \Rightarrow \infty \Leftarrow \infty$

8, $\left(\frac{3n-2}{3n+6}\right)^{5n+2} = \left(\frac{3n+6+(-8)}{3n+6}\right)^{5n+2} = \left(1+\frac{-8}{3n+6}\right)^{5n+2} = \left(\left(1+\frac{1}{\frac{3n+6}{-8}}\right)^{\frac{3n+6}{-8}}\right)^{\frac{(-8)\cdot(5n+2)}{3n+6}}$

$\frac{3n+6}{-8} \to -\infty$, $\to e$

$\frac{(5n+2)\cdot(-8)}{3n+6} = \frac{(-8)\cdot 5+\frac{2}{n}}{3+\frac{6}{n}} \Rightarrow \frac{-5\cdot 8}{3} = -\frac{40}{3}$ $n\to\infty$

$\frac{2}{n}\to 0$, $\frac{6}{n}\to 0$

$= e^{-\frac{40}{3}}$

(9) $\left(\frac{2n^2-3n+1}{2n^2-4n+3}\right)^{n+2} = \left(\frac{2n^2-4n+3+(n-2)}{2n^2-4n+3}\right) = 1+\frac{n-2}{2n^2-4n+3} =$

$\to \left(\left(1+\frac{1}{\frac{2n^2-4n+3}{n-2}}\right)^{\frac{2n^2-4n+3}{n-2}}\right)^{\frac{(n+2)(n-2)}{2n^2-4n+3}} \to e^{\frac{1}{2}} = \sqrt{e}$ $(n\to\infty)$

$\frac{2n^2-4n+3}{n-2} \to \infty$

$\frac{(n+2)(n-2)}{2n^2-4n+3} \frac{/n^2}{/n^2} \frac{\left(1+\frac{2}{n}\right)\cdot\left(1-\frac{2}{n}\right)}{2-\frac{4}{n}+\frac{3}{n^2}} \to \frac{1\cdot 1}{2} = \frac{1}{2}$

1°/1 F

2°/2 g K nem

4°/3 a b c D E M

7°/7 g nem

8/8 nem

9/9 mind

10°/10 ~~nem~~ nem g n o

11/11 h

~~13~~ P Q R nem

15/14 D E K M kell

16/16 Nem kell J K L

17/17 mind

17/18 nem kell $\sqrt[3]{}$