③ $\begin{matrix}2\\6\\5\end{matrix}$

wF

$$A=\begin{bmatrix}2&6&5\\-1&-4&1\\-1&-2&-3\end{bmatrix}$$ készítsük el a QR felb.

$$\tilde{Q}=\begin{bmatrix}2&0&1\\-1&-1&1\\-1&1&1\end{bmatrix}\qquad \tilde{R}=\begin{bmatrix}1&3&2\\0&1&-2\\0&0&1\end{bmatrix}$$

$$Q=\begin{bmatrix}\frac{2}{\sqrt6}&0&\frac{1}{\sqrt3}\\-\frac{1}{\sqrt6}&-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}\\-\frac{1}{\sqrt6}&\frac{1}{\sqrt2}&\frac{1}{\sqrt3}\end{bmatrix}\qquad R=\begin{bmatrix}\sqrt6&3\sqrt6&2\sqrt6\\0&\sqrt2&-2\sqrt2\\0&0&\sqrt3\end{bmatrix}$$

4, Határozzuk meg azt a Householder trf-t mely az $(1,1,1)^T$ vektort $r\cdot e_1=(r,0,0)^T$ alakra hozza, ellenőrzésként végezzük el a trf $\begin{bmatrix}1\\1\\1\end{bmatrix}\xrightarrow{H(v)}\begin{bmatrix}r\\0\\0\end{bmatrix}$

$$H(v)=I-2\,\underline{v}\cdot\underline{v}^T\qquad \underline{v}^T\cdot v=\langle \underline{v},\underline{v}\rangle\in\mathbb{R}$$

$$r=\|a\|_2\qquad \underline{v}=\frac{a-r\,\underline{e}_1}{\|a-r\,\underline{e}\|_2}$$

$$r=-\|\underline{a}\|_2=-\sqrt3\qquad \underline{a}-r\,\underline{e}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}-(-\sqrt3)\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1+\sqrt3\\1\\1\end{bmatrix}$$

$$v=\frac{1}{\left\|\begin{bmatrix}1+\sqrt3\\1\\1\end{bmatrix}\right\|_2}\begin{bmatrix}1+\sqrt3\\1\\1\end{bmatrix}=\frac{1}{\sqrt{(1+\sqrt3)^2+1+1}}\quad 1+2\sqrt3+3+2$$

$$2\cdot v\cdot v^T=2\cdot\frac{1}{6+2\sqrt3}\cdot\begin{bmatrix}1+\sqrt3\\1\\1\end{bmatrix}\cdot\left[1+\sqrt3,1,1\right]$$

$$\begin{bmatrix}1+\sqrt3\\1\\1\end{bmatrix}\left[1+\sqrt3\;\;1\;\;1\right]=\begin{bmatrix}(1+\sqrt3)^2&1+\sqrt3&1+\sqrt3\\1+\sqrt3&1&1\\1+\sqrt3&1&1\end{bmatrix}\qquad H_v=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\frac{1}{3+\sqrt3}\begin{bmatrix}(1+\sqrt3)^2&\cdots\\\vdots\\\end{bmatrix}$$

$$H(v)\cdot\underline{x}=\underline{x}-2\underline{v}\underbrace{(v^T\cdot x)}_{\in\mathbb{R}}=\underline{x}-2\cdot\langle\underline{v},\underline{x}\rangle\,\underline{v}$$

$$H(v)\underline{a}=\underline{a}-2\langle v,a\rangle\cdot v=\begin{bmatrix}1\\1\\1\end{bmatrix}-2\,\frac{1}{6+2\sqrt3}\cdot\left\langle\begin{bmatrix}1+\sqrt3\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\rangle\cdot\begin{bmatrix}1+\sqrt3\\1\\1\end{bmatrix}$$

$3+\sqrt3$

$$3+\sqrt3=1+\sqrt3+1+1$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix}-\frac{3+\sqrt3}{3+\sqrt3}\begin{bmatrix}1+\sqrt3\\1\\1\end{bmatrix}=\begin{bmatrix}-\sqrt3\\0\\0\end{bmatrix}$$

1, (5) oldjuk meg $A\underline{x}=\underline{b}$ LER-t Housh. trf-el

1

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \\ \underline{a_1}\ \ \underline{a_2}\ \ \underline{a_3} \end{array} \qquad \underline{b} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

1, lépés:

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \sigma_1 = \|a_1\|_2 = \sqrt{2}$$

$\sigma_1 \underline{e_1}$

$$\underline{a_1} - \sigma_1 \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \sqrt{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\|\underline{a}_1 - \sigma \underline{e}_1\|_2 = \sqrt{(1+\sqrt{2})^2+1} = \sqrt{4+2\sqrt{2}}$$

$$v_1 = \frac{1}{\sqrt{4+2\sqrt{2}}} \cdot \begin{bmatrix} 1+\sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

$$H_a v_1 = I - 2 v_1 v_1^T$$

$$A\underline{x} = b$$

$$HA\underline{x} = H\underline{b}$$

$$H(v_1) \cdot \underline{a}_1 = -\sqrt{2}\ \underline{e}_1 = \begin{bmatrix} -\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

$$H(v_1)\, \underline{a_2} = a_2 - 2 \underbrace{\overset{0}{v^T a_2}}_{\in \mathbb{R}} \cdot v_1 = a_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$H(v_1)\, a_3 = a_3 - 2(v_1^T, a_3) \cdot v_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - 2 \cdot \frac{1}{\underset{2+\sqrt{2}}{4+2\sqrt{2}}} \cdot \left\langle \begin{bmatrix} 1+\sqrt{2} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \frac{4+\sqrt{2}}{2+\sqrt{2}} \cdot \begin{bmatrix} 1+\sqrt{2} \\ 0 \\ 1 \end{bmatrix} =$$

$$2+\sqrt{2} \begin{bmatrix} 1+\sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1+2\sqrt{2} \\ 0 \\ 3-\sqrt{2} \end{bmatrix} = \begin{bmatrix} -2\sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix}$$

$$= \frac{4+\sqrt{2}}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{(4+\sqrt{2})(2-\sqrt{2})}{2-\sqrt{2}} = \frac{(4+\sqrt{2})(2-\sqrt{2})}{2} = \frac{8-2-2\sqrt{2}}{2} = \frac{6-2\sqrt{2}}{2} = (3-\sqrt{2})(1+\sqrt{2}) = \cancel{3-2}$$

$$= 3-2+2\sqrt{2} = 1+2\sqrt{2}$$

$$H(v_1) \cdot \underline{b} = \underline{b} - 2\langle v_1, \underline{b}\rangle v_1 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} - 2\, \frac{1}{\underset{2+\sqrt{2}}{4+2\sqrt{2}}} \left\langle \begin{bmatrix} 1+\sqrt{2} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \right\rangle \cdot \begin{bmatrix} 1+\sqrt{2} \\ 0 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} - \frac{6+2\sqrt{2}}{2+\sqrt{2}} \begin{bmatrix} 1+\sqrt{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2+3\sqrt{2} \\ 0 \\ 4-\sqrt{2} \end{bmatrix} = \begin{bmatrix} -3\sqrt{2} \\ 2 \\ +\sqrt{2} \end{bmatrix}$$

$$\frac{6+2\sqrt{2}}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{12-4-2\sqrt{2}}{2} = \frac{8-2\sqrt{2}}{2} = 4-\sqrt{2}$$

$$(4-\sqrt{2})(1+\sqrt{2}) = 4-2+3\sqrt{2} = 2+3\sqrt{2}$$

2)

$(3-\sqrt{2})(1+\sqrt{2})=3-2+2\sqrt{2}=1+2\sqrt{2}$

M

$$MA\underline{x}=\begin{bmatrix}-\sqrt{2} & 0 & -2\sqrt{2}\\ 0 & 2 & 0\\ 0 & 0 & \sqrt{2}\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=M\underline{b}\begin{bmatrix}-3\sqrt{2}\\ 2\\ \sqrt{2}\end{bmatrix}$$

$-\sqrt{2}x_1-2\sqrt{2}x_3=-3\sqrt{2}$ $\qquad$ $2x_2=2 \Rightarrow x_2=1$

$-\sqrt{2}x_1=-\sqrt{2}$ $\qquad$ $\sqrt{2}x_3=\sqrt{2} \quad x_3=1$

$x_1=1$

NORMÁK

Def legyen X vektortér $\mathbb{K}$ felett ($\mathbb{K}=\mathbb{R} \vee \mathbb{C}$)

A $\| \; \|: X \to \mathbb{R}$ fv. (vektor)norma ha

- $\|x\| \geq 0 \quad \forall \underline{x} \in X$
- $\|x\| = 0 \iff \underline{x}=0$
- $\|\lambda \underline{x}\| = |\lambda| \|\underline{x}\| \quad \forall \lambda \in \mathbb{K} \; \forall \underline{x} \in X$
- $\|x+y\| \leq \|x\| + \|y\|$ $\qquad$ Δ-egyenlőtlenség

Áll: euklideszi térben $(V \langle\,\rangle)$ norma értelmezhető a skalár-szorzat szerint

$\|x\| = \sqrt{\langle x,x\rangle}$

$|\langle x,y\rangle| = \|x\| \|y\| \quad \forall \underline{x}, \underline{y} \in V$

Áll: A következő formulák $\mathbb{R}^n$-ben vektornormát definiálnak

$$\|\underline{x}\|_1 = \sum_{i=1}^{n} |x|$$

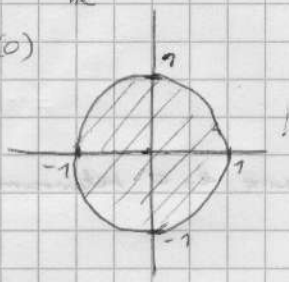
$$\|\underline{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

$$\|x\|_\infty = \max_{i=1}^{n} |x_i|$$

3 $K_1(0)$

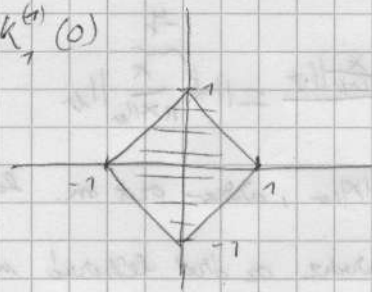
$\mathbb{R}^2$

$K_1^{(2)}(0)$

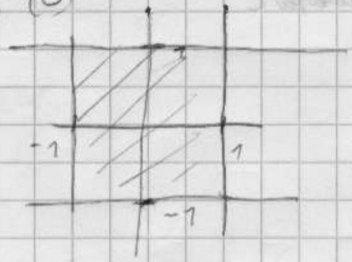


!

$K_1^{(1)}(0)$



$K_1^{(\infty)}(0)$



Def A $\|\ \|_A$ és $\|\ \|_B$ vektornormák ekvivalensek ha $\exists c_1, c_2 > 0$: $c_1\|x\|_B \le \|x\|_A \le$
$\le c_2\|x\|_B \quad \forall x \in \mathbb{K}^n$

Áll: $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$
$\|x\|_\infty \le \|x\|_2 \le \|x\|_\infty \qquad \forall x \in \mathbb{K}^n$
$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$

Tétel: Véges dimenziós vektortérben bármely két vektornorma ekvivalens

Def Az $x^{(k)} \in \mathbb{K}^n$ konvergens ha $\exists x \in \mathbb{K}$
$\lim_{k\to\infty} \|x^{(k)} - x\| = 0$

Def: A $\|\ \| : \mathbb{K}^{n\times n} \to \mathbb{R}$ leképezést mátrixnormának nevezzük ha a következők teljesülnek

1. $\|A\| \ge 0 \quad \forall A \in \mathbb{K}^{n\times n}$
2. $\|A\| = 0 \iff A = 0$
3. $\|\lambda A\| = |\lambda|\,\|A\| \quad \forall \lambda \in \mathbb{K}\ \forall A \in \mathbb{K}^{n\times n}$
4. $\|A+B\| \le \|A\| + \|B\| \quad \forall A, B \in \mathbb{K}^{n\times n}$
5. $\|A\cdot B\| \le \|A\|\cdot\|B\| \quad \forall A, B \in \mathbb{K}^{n\times n}$

Tétel Mátrixnorma konstrukciója vektornormából

Legyen $\|\ \|_v$ egy vektornorma ekkor $\forall A \in \mathbb{K}^{n\times n}$

$\|A\| = \sup_{x\neq 0} \frac{\|A\cdot x\|_v}{\|x\|_v}$ ez a vele megegyező mátrixnorma

4, Def $\|A\| = \sup\limits_{\|x\|_v=1} \|A\underline{x}\|_v$

Biz $\frac{\|Ax\|_v}{\|x\|_v} = \frac{\|x\|_v \|A\frac{x}{\|x\|}\|_v}{\|x\|_v} = \|A \overbrace{\frac{x}{\|x\|_v}}^{y}\|_v$

Def Ha $\|Ax\|_v \leq \|A\|\cdot\|x\|_v$, akkor azt m. hogy a mátrixnorma és a vektornorma illeszkedik

Áll indukált mátrixnormák illeszkednek az őket létrehozó vektornormákhoz

Tétel: Az $\|x\|_1$ $\|x\|_2$ $\|x\|_\infty$ vektornormák a köv. mátrixokon hadaradják

$\|A\|_1 = \max\limits_{j=1}^{n} \sum\limits_{i=1}^{n} |a_{ij}|$ oszlopnorma

$\|A\|_\infty = \max\limits_{i=1}^{n} \sum\limits_{j=1}^{n} |a_{ij}|$ sornorma

$\|A\|_2 = \max\limits_{i=1}^{n} (\lambda(A^*A))^{1/2} = \underbrace{\varrho}_{\text{spektrálsugár}}(A^*A)^{1/2}$ Spektrálnorma

Def $\|A\|_F = \left(\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{n} |a_{ij}|^2\right)^{1/2}$ Frobenius-norma

LER érzékenysége (kondicionáltság)

Vizsgáljuk meg hogyan vált a LER m.o. ha a mátrixot V a felbecsült változtatjuk

$A\underline{x} = \underline{b}$ helyett
1. $A(\underline{x}+\Delta x) = \underline{b} + \Delta b$
2. $(A+\Delta A)(\underline{x}+\Delta x) = \underline{b}$
3. a kettő együtt

Tétel Ha A-t invertálom és $\underline{b} \neq \underline{0}$ akkor az 1, esetén $\frac{1}{\|A\|\|A^{-1}\|} \cdot \frac{\|\Delta b\|}{\|b\|} \leq \frac{\|\Delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\| \frac{\|\Delta b\|}{\|b\|}$

Tétel Ha A invertálható $\underline{b} \neq \underline{0}$ és $\|\Delta A\|\cdot\|A^{-1}\| < 1$ akkor a 2. esetén

$$\frac{\|\Delta \underline{x}\|}{\|x\|} \leq \frac{\|A\|\|A^{-1}\|}{1-\|\Delta A\|\|A^{-1}\|} \cdot \frac{\|\Delta A\|}{\|A\|}$$

Def Az $\|A\|\|A^{-1}\| = \mathrm{cond}(A)$ mennyiséget kondíciószámnak nevezzük

Tétel kondíciószám tulajdonsága

1, $\forall c \in \mathbb{K} \setminus \{0\}$ $\mathrm{cond}(cA) = \mathrm{cond}(A)$

2, indukált mátrixnorma esetén $\mathrm{cond}(A) \geq 1$

3, $Q \in \mathbb{K}^{n\times n}$ unitér (ortogonális) $\mathrm{cond}_2(Q) = 1$

4 A szimmetrikus és pozitív adef $\mathrm{cond}_2(A) = \frac{\max \lambda_i(A)}{\min \lambda_i(A)}$

5, A szimm $\mathrm{cond}_2(A) = \frac{\max|\lambda_i(A)|}{\min|\lambda_i(A)|}$

1, Írjuk az $\| \|_1$ vektornorma által indukált mátrixnormát

$$\|A\|_1 = \sup_{x \neq 0} \frac{\|A\underline{x}\|_1}{\|x\|_1} = \max_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A x\|_1 = \sum_{i=1}^{n} \Big| \underbrace{\sum_{j=1}^{n} a_{ij} x_j}_{(Ax)_i} \Big| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{n} |x_j| \sum_{i=1}^{n} |a_{ij}| \leq$$

$$\leq \sum_{j=1}^{n} |x_j| \max \sum_{j=1}^{n} |a_{ij}| = \max_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}| \cdot \sum_{j=1}^{n} |x_j|$$

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|$$

$$\frac{\|Ax\|_1}{\|x\|} \leq \max_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}|$$

legyen p amelyre $\sum_{i=1}^{n} |a_{ip}| = \max_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}|$

$$\underline{x} := e_{\underline{p}} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} p \quad \|A\underline{x}\|_1 = \max_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|x\|_1 = 1$$

3, adjuk meg az $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ mátrix $\| \|_1$ $\| \|_2$ $\| \|_\infty$ $\| \|$ normára vonatkozó kondíciószámát

$$\text{cond}_1 A = \|A\|_1 \cdot \|A^{-1}\|_1 = \underline{3 \cdot 1 = 3}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = -\frac{1}{3} \overbrace{\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}}^{B} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$\det A = 1 - 4 = -3$ $\qquad \|A^{-1}\| = \left|-\frac{1}{3}\right| \cdot \|B\| = \frac{1}{3} \max\{3, 3\} = 1$

$\|A\|_1 \max\{3, 3\} = 3$ -> oszlop össz. legnagyobb eleme

$\|A^{-1}\|_1 = 1$

$\|A\|_\infty = \max\{3, 3\} = 3$ -> sorok összeg legnagyobb eleme

$\|A^{-1}\|_\infty = \frac{1}{3} \max\{3, 3\} = 1$ $\qquad \text{cond}_\infty A = 3 \cdot 1 = 3$

$\|A\|_F = (1 + 4 + 4 + 1)^{1/2} = \sqrt{10}$ $\qquad \text{cond}_F A = \frac{1}{3} \sqrt{10} \sqrt{10} = \frac{10}{3}$

$\|A^{-1}\|_F = \frac{1}{3}(1 + 4 + 4 + 1)^{1/2} = \frac{1}{3}\sqrt{10}$

$\|A\|_2 = \left[\rho(A^* A)\right]^{1/2}$

Ha A normális

$\|A\|_2 = \rho(A) = 3$

$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0$ $\quad \lambda_1 = 3$

$(1-\lambda)^2 = 4$ $\quad \lambda_2 = 1$

$1-\lambda = \pm 2$ $\quad \rho(A^{-1}) = \frac{1}{\min|(\lambda_1)|} = 1$

$1 \pm 2 = \lambda$

$\text{cond}_2 A = 3 \cdot 1 = \underline{\underline{3}}$

$$A = \begin{array}{cc} \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} & \begin{array}{c} 3 \\ 10 \end{array} \\ \begin{array}{cc} 4 & 9 \end{array} & \end{array}$$ $\text{cond}_1 = A$ HF

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\|A_1\|_1 = \max\{4, 9\} = 9$ $\quad \det A = 1$

$\|A\|_\infty = \max\{3, 10\} = 10$

$A^{-1} = \frac{1}{1} \cdot \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

~~$\|A\|_1 = \max\{10, 3\}$~~

$\|A^{-1}\|_1 = \max\{10, 3\} = 10$ $\quad \text{cond}_1 A = 9 \cdot 10 = 90$

$\|A^{-1}\|_\infty = \max\{9, 4\} = 9$ $\quad \text{cond}_\infty A = 10 \cdot 9 = 90$

$\|A_2\| = \left[\rho(A^* A)\right]^{\frac{1}{2}} =$

$A^* A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 23 \\ 23 & 53 \end{bmatrix}$

~~$\begin{vmatrix} 10-\lambda & 23 \\ 23 & 53-\lambda \end{vmatrix} = (10-\lambda)^2 - 23^2 = 0$~~

~~$(10-\lambda)^2 = 23^2$~~

~~$10-\lambda = \pm 23$~~

~~$10 \pm 23 = \lambda$~~

$\begin{bmatrix} 10-\lambda & 23 \\ 23 & 53-\lambda \end{bmatrix} = (10-\lambda)(53-\lambda) - 23^2 = 530 - 63\lambda + \lambda^2 - 529 = 0$

$\lambda^2 - 63\lambda + 1 = 0$ $\quad \lambda_1 = 62{,}98$

$\lambda_{1,2} = \frac{63 \pm \sqrt{3969 - 4}}{2}$ $\quad \lambda_2 = 0{,}02$

$\|A\|_2 = 62{,}98$

$\|A^{-1}\|_2 = ?$

$$A=\begin{bmatrix}7 & 9 & -4\\ -8 & 9 & -7\\ 4 & -6 & 6\end{bmatrix} \qquad \text{cond}_1 A \qquad \text{cond}_\infty A$$

$$\left[\begin{array}{ccc|ccc}7 & 9 & -4 & 1 & 0 & 0\\ -8 & 9 & -7 & 0 & 1 & 0\\ 4 & -6 & 6 & 0 & 0 & 1\end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc}7 & 9 & -4 & 1 & 0 & 0\\ 0 & \frac{135}{7} & -\frac{81}{7} & \frac{8}{7} & 1 & 0\\ 0 & -\frac{78}{7} & \frac{58}{7} & -\frac{4}{7} & 0 & 1\end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc}7 & 9 & -4 & 1 & 0 & 0\\ 0 & \frac{135}{7} & -\frac{81}{7} & \frac{8}{7} & 1 & 0\\ 0 & 0 & \frac{8}{5} & \frac{4}{45} & \frac{26}{45} & 1\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc}7 & 9 & 0 & \frac{11}{9} & \frac{13}{9} & \frac{5}{2}\\ 0 & \frac{135}{7} & 0 & \frac{25}{14} & \frac{145}{28} & \frac{405}{56}\\ 0 & 0 & 1 & \frac{1}{18} & \frac{13}{36} & \frac{5}{8}\end{array}\right]$$

$$\frac{78}{\cancel{7}}\cdot\frac{\cancel{7}}{135}=\frac{26}{45}$$

$$\left[\begin{array}{ccc|ccc}7 & 0 & 0 & \frac{7}{18} & -\frac{35}{36} & -\frac{7}{8}\\ 0 & 1 & 0 & \frac{5}{54} & \frac{29}{108} & \frac{3}{8}\\ 0 & 0 & 1 & \frac{1}{8} & \frac{13}{36} & \frac{5}{8}\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix}\frac{1}{18} & -\frac{5}{36} & -\frac{1}{8}\\ \frac{5}{54} & \frac{29}{108} & \frac{3}{8}\\ \frac{1}{18} & \frac{13}{36} & \frac{5}{8}\end{matrix}}_{A^{-1}}\\ 0 & 0 & 1 & \end{array}\right]$$

$$A=\begin{bmatrix}7 & 9 & -4\\ -8 & 9 & -7\\ 4 & -6 & 6\end{bmatrix}\begin{matrix}20\\24\\16\end{matrix}$$

$$19 \quad 24 \quad 17$$

~~3+5+3~~

$\|A\|_1 = 24$

$\|A\|_\infty = 24$

$\text{cond}_1 A = 24\cdot\frac{9}{8} = 27$

$\text{cond}_\infty A = \frac{25}{\cancel{24}} = 25$

$\frac{3+5+3}{54} = \frac{11}{54}$ $\qquad$ $\frac{4+10+9}{72} = \frac{23}{72}$

$\frac{15+29+39}{108} = \frac{83}{108}$ $\qquad$ $\frac{20+58+81}{216} = \frac{159}{216}$

$\|A^{-1}\|_1 = \frac{9}{8}$

$\frac{1+3+5}{8} = \frac{9}{8}$ $\qquad$ $\frac{4+26+45}{72} = \frac{75}{72}$

$\|A^{-1}\|_\infty = \frac{75}{72} = \frac{25}{24}$

1, Oldjuk meg az $Ax=$

$$A=\begin{bmatrix}-5 & -6 & 8\\ 0 & 3 & 3\\ 3 & -5 & 8\end{bmatrix} \qquad b=\begin{bmatrix}7\\ 15\\ 17\end{bmatrix}$$

gauss eliminációval

$$\left[\begin{array}{ccc|c}-5 & -6 & 8 & 7\\ 0 & 3 & 3 & 15\\ 3 & -5 & 8 & 17\end{array}\right] \quad \left[\begin{array}{ccc|c}-5 & -6 & 8 & 7\\ 0 & 3 & 3 & 15\\ 0 & -\frac{43}{5} & \frac{64}{5} & \frac{106}{5}\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}-5 & -8 & 8 & 7\\ 0 & 3 & 3 & 15\\ 0 & 0 & \frac{107}{5} & \frac{321}{5}\end{array}\right]$$

$+ -6\cdot\frac{3}{5}$

$\frac{1}{3}\cdot\frac{43}{5}$

$\cancel{0\ 0}$

$$\left[\begin{array}{ccc|c}-5 & -6 & 0 & -17\\ 0 & 3 & 0 & 6\\ 0 & 0 & 1 & 3\end{array}\right] = \left[\begin{array}{ccc|c}-5 & 0 & 0 & -5\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 3\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 3\end{array}\right] \qquad x=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

2, Adjuk meg

$$B\begin{bmatrix}-9 & 5 & -2\\ 1 & -2 & -4\\ 8 & 3 & 3\end{bmatrix}$$

LDU felbontását

$$B=\tilde{L}\tilde{U}=\tilde{L}(DD^{-1})\tilde{U}=\tilde{L}D\underbrace{(D^{-1}\tilde{U})}_{U}$$

$$B=\tilde{L}\tilde{U} \qquad D\begin{bmatrix}u_1 & 0 & 0\\ 0 & u_4 & 0\\ 0 & 0 & u_6\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ l_1 & 1 & 0\\ l_2 & l_3 & 1\end{bmatrix}\begin{bmatrix}u_1 & u_2 & u_3\\ 0 & u_4 & u_5\\ 0 & 0 & u_6\end{bmatrix}$$

$L=\tilde{L}$

$U=D^{-1}\tilde{U}$

D

$$\begin{array}{c|c} & \begin{bmatrix}u_1 & u_2 & u_3\\ 0 & u_4 & u_5\\ 0 & 0 & u_6\end{bmatrix}\\ \hline \begin{bmatrix}1 & 0 & 0\\ l_1 & 1 & 0\\ l_2 & l_3 & 1\end{bmatrix} & \begin{matrix}-9 & 5 & -2\\ 1 & -2 & -4\\ 8 & 3 & 3\end{matrix}\end{array}$$

$1\cdot u_1 = -9 \Rightarrow u_1=-9$

$u_2 = 5$

$u_3 = -2$

$l_1\cdot u_1 = 1$

$l_1\cdot(-9)=1 \Rightarrow l_1 = -\frac{1}{9}$

$u_1\, l_2 = 8 \Rightarrow l_2 = -\frac{8}{9}$

$\underbrace{l_1\cdot u_2}_{-\frac{5}{9}} + u_4 = -2 \qquad u_4 = -2 + \frac{5}{9} = -\frac{13}{9}$

$\underbrace{l_1\cdot u_3}_{\frac{2}{9}} + u_5 = -4 \Rightarrow u_5 = -\frac{36}{9} - \frac{2}{9} = -\frac{38}{9}$

$l_2\cdot u_2 + l_3\cdot u_4 = 3$

$-\frac{8}{9}\cdot 5 + l_3\cdot\left(-\frac{13}{9}\right) = 3$

$l_3\cdot -\frac{13}{9} = \frac{27}{9} + \frac{40}{9} = \frac{67}{9}$

$l_3 = -\frac{67}{13}$

$l_2\cdot u_3 + l_3\cdot u_5 + u_6$

$u_6 = 3 - \left(-\frac{8}{9}\right)(-2) - \left(\frac{67}{13}\right)\left(-\frac{38}{9}\right) = \frac{-267}{13}$

$\tilde{L}O$

$$\tilde{L} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{9} & 1 & 0 \\ -\frac{8}{9} & -\frac{57}{13} & 1 \end{bmatrix} \qquad \tilde{U} = \begin{bmatrix} -9 & 5 & -2 \\ 0 & -\frac{13}{9} & -\frac{38}{9} \\ 0 & 0 & -\frac{267}{13} \end{bmatrix}$$

$$D = \begin{bmatrix} -9 & 0 & 0 \\ 0 & -\frac{13}{9} & 0 \\ 0 & 0 & -\frac{267}{13} \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} -\frac{1}{9} & 0 & 0 \\ 0 & -\frac{9}{13} & 0 \\ 0 & 0 & -\frac{13}{267} \end{bmatrix}$$

$L = \tilde{L}$

D $\qquad U = D^{-1}\tilde{U} = \begin{bmatrix} -\frac{1}{9} & 0 & 0 \\ 0 & -\frac{9}{13} & 0 \\ 0 & 0 & -\frac{13}{267} \end{bmatrix} \cdot \begin{bmatrix} -9 & 5 & -2 \\ 0 & -\frac{13}{9} & -\frac{38}{9} \\ 0 & 0 & -\frac{267}{13} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{5}{9} & \frac{2}{9} \\ 0 & 1 & \frac{38}{13} \\ 0 & 0 & 1 \end{bmatrix}$

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3) Tanult módszerrel QR felbontás: (Gram Smith) (Householder)

$$C \begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \qquad Q = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{2} & 0 & 5/\sqrt{2} \\ 0 & 2 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$q_1 = \frac{1}{r_{11}} c_1 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$r_{11} = \|c_1\|_2 = \sqrt{2}$

$$q_2 = \frac{1}{r_{22}}(c_2 - r_{12}\, q_1) = \frac{1}{r_{22}} c_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$(r_{12} = \langle c_2, q_1 \rangle = 0) \quad r_{22} = \|c_2\|_2 = 2$

$$q_3 = \frac{1}{r_{33}} \cdot (c_3 - r_{13} \cdot q_1 + r_{23} q_2) = \frac{1}{r_{33}}\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} - \frac{5}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) = \frac{1}{r_{33}} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} \qquad r_{33} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$$

$r_{13} = \langle c_3, q_1 \rangle = \left\langle \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle = \frac{1}{\sqrt{2}} \left\langle \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle \frac{5}{2}$ $\qquad q_3 = \sqrt{2} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$r_{13} = \langle c_3\ q_1 \rangle = \frac{5}{\sqrt{2}}$

$r_{23} = \langle c_3 \cdot q_2 \rangle = 0$

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4) Adjuk meg a

$$D = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$ kondicionálását

$\text{cond}_1 D = ? \qquad \text{cond}_2 D = ?$

$3 \cdot 3 = 9 \quad 1 \cdot 2 = 2 \quad 9 - 2 = 7 = \frac{1}{7}$

$\text{cond}_v D = \|D\|_v \cdot \|D^{-1}\|_v$

$$D^{-1} = \frac{1}{\det_D} \cdot \begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix} = -\frac{1}{7} \cdot \begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix}$$

$\|D^{-1}\|_1 = \left|\frac{1}{7}\right| \max\{5, 4\} = \frac{5}{7}$

$\text{cond}_1 D = 5 \cdot \frac{5}{7} = \frac{25}{7}$

$\|D\|_2 = \varrho(D) = \frac{3+\sqrt{37}}{2} \approx 4{,}5414$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 9 = \lambda^2 - 3\lambda - 7 = 0$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{37}}{2}$$

$$\|D^{-1}\|_2 = \frac{1}{\min|\lambda_i|} = \frac{1}{\left|\frac{3-\sqrt{37}}{2}\right|} = \left|\frac{2}{3-\sqrt{37}}\right| = \left|\frac{2 \cdot (3+\sqrt{37})}{9-37}\right| = \frac{3+\sqrt{37}}{14} \approx 0{,}6488$$

$$\text{cond}_2 D = \frac{3+\sqrt{37}}{2} \cdot \frac{3+\sqrt{37}}{14} = \frac{9+6\sqrt{37}+37}{28} = \frac{46+6\sqrt{37}}{28} = \frac{23+3\sqrt{37}}{14} \approx 2{,}9463$$

$\parallel$

$2{,}9465$

5, Adjuk meg a kondíciószámokat

$$E = \begin{bmatrix} -3 & -3 & -3 \\ -4 & -5 & 0 \\ 4 & 3 & 0 \end{bmatrix}$$

$\text{cond}_F E = ?$

$\text{cond}_\infty E = ?$

$+\frac{4}{3}$ [illegible]

$$\left[\begin{array}{ccc|ccc} -3 & -3 & -3 & 1 & 0 & 0 \\ -4 & -5 & 0 & 0 & 1 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} -3 & -3 & -3 & 1 & 0 & 0 \\ 0 & -1 & 4 & \frac{4}{3} & 1 & 0 \\ 0 & -1 & -4 & \frac{4}{3} & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} -3 & -3 & -3 & 1 & 0 & 0 \\ 0 & -1 & 4 & -\frac{4}{3} & 1 & 0 \\ 0 & 0 & -8 & \frac{8}{3} & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} -3 & -3 & 0 & 0 & \frac{3}{8} & -\frac{3}{8} \\ 0 & -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{8} & -\frac{1}{8} \end{array}\right] \Rightarrow$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} -3 & 0 & 0 & 0 & -\frac{9}{8} & -\frac{15}{8} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{8} & -\frac{1}{8} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{3}{8} & \frac{5}{8} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{8} & -\frac{1}{8} \end{array}\right]$$

$$\underbrace{\quad\quad\quad}_{E^{-1}}$$

$$\|E\|_F = \sqrt{9+9+9+16+25+16+9} = \sqrt{93} \qquad \|E^{-1}\|_F \sqrt{\frac{169}{144}} = \frac{13}{12}$$

$$\text{cond}_F E = \sqrt{93} \cdot \frac{13}{12} \approx 10{,}4473$$

$$\|E\|_\infty = \max\{9, 9, 7\} = 9 \qquad \|E^{-1}\|_\infty = \{1, 1, \tfrac{14}{8}\} = \frac{14}{8}$$

$$\text{cond}_\infty E = 9 \cdot \frac{7}{4} = \frac{63}{4} = 15{,}75$$