$Ax = b \iff x = Bx + c$

$$x^{(k+1)} = Bx^{(k)} + c \qquad x^0 \in \mathbb{R}^n$$

GYAK

Iterációk $\quad x^{(k+1)} = Bx^{(k)} + c$

$$B = \begin{bmatrix} 0{,}4 & 0{,}1 & 0 \\ 0{,}1 & 0{,}3 & 0{,}2 \\ 0 & 0{,}1 & 0{,}4 \end{bmatrix} \qquad c \begin{bmatrix} 0{,}1 \\ 0{,}5 \\ 0{,}1 \end{bmatrix}$$

$\|B_1\| = \|B\|_\infty = 0{,}6 = q < 1 \Rightarrow \forall x^0$ konvergens

$$\|x^k - x^*\|_1 \le \frac{q^k}{1-q} \| x_1^{(1)} - x_0^{(0)} \|_1$$

$x^0 = 0$ hány lépés garantált 10^{-3}-as pontosságot $\qquad x^{(1)} = Bx^{(0)} + c$

$$\|x^{(k)} - x^*\|_1 \le \frac{0{,}6^k}{0{,}4} \cdot 0{,}7 < 10^{-3} \qquad \|x^{(1)} - x^{(0)}\|_1 = \|c\|_1 = 0{,}7$$

$$\left(\frac{3}{5}\right)^k \cdot \frac{7}{4} < 10^{-3}$$

$$\frac{7}{4} \cdot 1000 < \left(\frac{5}{3}\right)^k$$

$$\lg\left(\frac{7}{4} \cdot 1000\right) < k \lg \frac{5}{3}$$

$$14{,}6 \approx \frac{\lg \frac{7}{4} \cdot 1000}{\lg \frac{5}{3}} < k$$

$$x^* = Bx^* + c$$

$$\underbrace{(I - B)}_{A} x^* = \underset{\sim}{\underbrace{c}_{b}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0{,}4 & 0{,}1 & 0 \\ 0{,}1 & 0{,}3 & 0{,}2 \\ 0 & 0{,}1 & 0{,}4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0{,}6 & -0{,}1 & 0 \\ -0{,}1 & 0{,}7 & -0{,}2 \\ 0 & -0{,}1 & 0{,}6 \end{bmatrix}}_{A}$$

2. Konvergál-e az

$$A = \begin{array}{c} \\ L \\ \\ \end{array}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & -1 \\ -2 & -2 & 1 \end{bmatrix}$$

mátrixra felírt Jacobi eljárás

$$A = L + D + U$$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$Ax = b$

$(L + D + U)x = b$

$(L + U)x + Dx = b$

$Dx = -(L + U)x + b$

$x = -D^{-1}(L + U)x + D^{-1}b$

$$B_J = -D^{-1}(L+U) = -D^{-1} \cdot \begin{bmatrix}0 & -2 & 2\\ -1 & 0 & -1\\ -2 & -2 & 0\end{bmatrix} = \begin{bmatrix}0 & 2 & -2\\ 1 & 0 & 1\\ 2 & 2 & 0\end{bmatrix}$$

$$\cancel{\begin{bmatrix}0 & -2 & 2\\ -1 & 0 & -1\\ -2 & -2 & 0\end{bmatrix}} \cdot$$

$\|B_J\| < 1$ nem telj. $\qquad \varrho(B) < 1$

$\|B\|_1 = 4$

$\|B\|_\infty = 4$

$\|B\|_F = \sqrt{18}$

$$\det(B_J - \lambda I) = 0$$

$$\begin{vmatrix}-\lambda & 2 & -2\\ 1 & -\lambda & 1\\ 2 & 2 & -\lambda\end{vmatrix} = -\lambda(\lambda^2 - 2) - 2(-\lambda - 2) - 2(2 + 2\lambda) =$$

$$-\lambda^3 + 2\lambda + 2\lambda + 4 - 4 - 4\lambda = -\lambda^3 = 0$$

$x_{1,2,3} = 0$

$\varrho(B) = 0 < 1$

$\Downarrow$

iteráció

$\forall x_0 \in \mathbb{R}^n$ konvergens

3) Legyen $A = \begin{bmatrix}4 & -1 & 0\\ -1 & 4 & -1\\ 0 & -1 & 4\end{bmatrix}$ $\qquad \underline{b} = \begin{bmatrix}2\\ 6\\ 2\end{bmatrix}$ $\qquad A\underline{x} = \underline{b}$

- $J(1)$ $\qquad J(\omega)$
- Konvergens-e?
- Adjuk meg hány lépés a $J(1)$ $\underline{x}^{(0)} = \underline{0}$ kezdővel éri el a 10^{-2}-os pontosságot
- Írjuk fel az iteráció első 3 elemét
- Mely ω-ra konvergens $J(\omega)$ melyik ω-ra optimális.

$$B_J = -D^{-1}(L+U) = \underbrace{\begin{bmatrix}1/4 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & 0 & 1/4\end{bmatrix}}_{D^{-1}} \cdot \begin{bmatrix}0 & -1 & 0\\ -1 & 0 & -1\\ 0 & -1 & 0\end{bmatrix} = -\begin{bmatrix}0 & -\frac{1}{4} & 0\\ -\frac{1}{4} & 0 & -\frac{1}{4}\\ 0 & -\frac{1}{4} & 0\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{4} & 0\\ \frac{1}{4} & 0 & \frac{1}{4}\\ 0 & \frac{1}{4} & 0\end{bmatrix} = B_J$$

$$\|B_J\|_1 = \|B_J\|_\infty = \frac{1}{2} = q < 1$$

$$\left(\varrho(B_J) = \|B_J\|_2 = \frac{1}{\sqrt{8}} = q < 1 \right)$$

$$x^{(1)} = B_J \underset{\underset{0}{\sim}}{x^0} + c_J = c_J = D^{-1} b = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\|x^{(2)} - x^*\|_\infty \le \frac{q^2}{1-q} \| x^{(1)} - x^{(0)} \|_\infty -$$

$$\|x^{(1)} - x^{(0)}\|_\infty = \|c_J\|_\infty = \frac{(\frac{1}{2})^2}{\frac{1}{2}} \cdot \frac{3}{2} = 3 \cdot \left(\frac{1}{2}\right)^2 < 10^{-2}$$

$$3 \cdot 100 < 2^2$$

$$2^8 < 300 < 2^9 \quad 9 \le 2$$

$$x^2 = B_J \cdot x^{(1)} + c_J = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 3/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 3/2 \\ 1/2 \end{bmatrix}$$

$$x^2 = \begin{bmatrix} 3/8 \\ 2/8 \\ 3/8 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 7/8 \\ 14/8 \\ 7/8 \end{bmatrix} = \begin{bmatrix} 7/8 \\ 7/4 \\ 7/8 \end{bmatrix}$$

$$x^{(3)} = B_J \cdot x^{(2)} \cdot c_J = \begin{bmatrix} 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 \\ 0 & 1/4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 7/8 \\ 7/4 \\ 7/8 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 3/2 \\ 1/2 \end{bmatrix} =$$

$$= \begin{bmatrix} 7/16 \\ 7/16 \\ 7/16 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 15/16 \\ 31/16 \\ 15/16 \end{bmatrix}$$

$$Dx = -(L+U)x + b$$

$$Dx = Dx$$

$$B_{J}(\omega) = (1-\omega) I - \omega D^{-1} (L+U) = (1-\omega) \cdot I + \omega \cdot \begin{bmatrix} 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 \\ 0 & 1/4 & 0 \end{bmatrix} = \begin{matrix} 1-\omega & \omega/4 & 0 \\ \omega/4 & \cancel{1-\omega} & \omega/4 \\ \cancel{0} & \omega/4 & 1-\omega \end{matrix}$$

$$\begin{bmatrix} 1-\omega & 0 & 0 \\ 0 & 1-\omega & 0 \\ 0 & 0 & 1-\omega \end{bmatrix}$$

$$\mathrm{Det}(B_J(\omega) - \lambda I) = \begin{bmatrix} 1-\omega-\lambda & \omega/4 & 0 \\ \omega/4 & 1-\omega-\lambda & \omega/4 \\ 0 & \omega/4 & 1-\omega-\lambda \end{bmatrix} = (1-\omega-\lambda) \cdot \left((1-\omega-\lambda)^2 - \frac{\omega^2}{16}\right) - \frac{\omega}{4} \cdot \frac{\omega}{4} (1-\omega-\lambda) =$$

$$= (1-\omega-\lambda) \cdot \left((1-\omega-\lambda)^2 - \frac{\omega^2}{8}\right) = \cancel{(1-\omega-\lambda)}$$

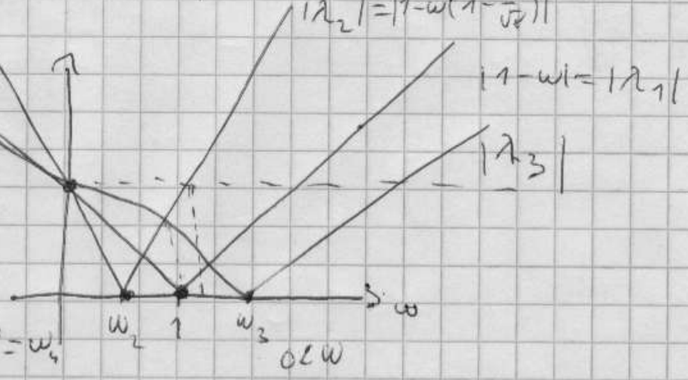
$$= (1-\omega-\lambda)\left(1-\omega-\lambda+\frac{\omega}{\sqrt{8}}\right) \cdot \left(1-\omega-\lambda-\frac{\omega}{\sqrt{8}}\right) = 0$$

λ_1 $\quad 1-\omega-\lambda = 0 \Rightarrow \lambda_1 = 1-\omega$

λ_2 $\quad 1-\omega-\lambda+\frac{\omega}{\sqrt{2}} = 0 \Rightarrow \lambda_2 = 1-\omega\left(1-\frac{1}{\sqrt{8}}\right)$

λ_3 $\quad 1-\omega-\lambda-\frac{\omega}{\sqrt{8}} = 0 \Rightarrow \lambda_3 = 1-\omega\left(1+\frac{1}{\sqrt{8}}\right)$

$$\rho(B_J\,\omega) = \max\{|\lambda_1|;\ |\lambda_2|;\ |\lambda_3|\}$$



$|\lambda_1| = 1$ $\quad 2 = \omega(1-\frac{1}{\sqrt{8}}) = \frac{2}{1-\sqrt{8}} = \omega_4$

$1-\omega(1-\frac{1}{\sqrt{8}}) = -1$

$\cancel{2 = \omega\, 1 - \frac{1}{\sqrt{8}} = \omega = \frac{2}{1}}$

$|1-\omega(1-\frac{1}{\sqrt{8}})|$ $\qquad \omega(1-\frac{1}{\sqrt{8}}) = 1 \quad \frac{\sqrt{8}-1}{\sqrt{8}} = \frac{8-\sqrt{8}}{8-1} = \frac{8}{7} - \frac{\sqrt{8}}{7}$

$|\lambda_2| = |\lambda_3|$

$1-\omega(1-\frac{1}{\sqrt{8}}) = 1-\omega(1+\frac{1}{\sqrt{8}})$

$2 = \omega\cancel{\frac{1}{\sqrt{8}}}) + 1 - \cancel{\frac{1}{\sqrt{8}}}) = 2\omega = \omega = 1$

FMAK

1.

$$A = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 1 & -1 \\ -2 & -2 & 1 \end{bmatrix}$$ Konvergal-e a mátrix szerint $S(1)(G-S)$

$A = L + D + U$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$A\underline{x} = \underline{b}$

$(L+D+U)\underline{x} = \underline{b}$

$(L+D)\underline{x} = -U\underline{x} + \underline{b}$

$$\underline{x} = \underbrace{-(L+D)^{-1} U}_{B_S}\underline{x} + \underbrace{(L+D)^{-1}}_{C_S}\underline{b}$$

$$B_S = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 2 & -1 & 2 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 4 & 2 & 1 \end{array}\right] \Rightarrow \underbrace{\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & -2 & -1 \end{matrix}}_{-[(L+D)]^{-1}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ -4 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 2 & 2 \\ 0 & 2 & 3 \\ 0 & 8 & 10 \end{bmatrix}}_{B_S}$$

$\|B_S\|_1 = 15$

$\|B_S\|_\infty = 18$

$\|B_S\|_F = \sqrt{185}$

$\|B_S\|_2$

$S(B_S) = \max\{|\lambda| \; k_{B_S}(\lambda_i) = 0\}\; 6 + 2\sqrt{10} > 1$ Nem minden x_0-ra konv.

$$\det(B_S - \lambda I) = \begin{vmatrix} -\lambda & 2 & 2 \\ 0 & 2-\lambda & 3 \\ 0 & 8 & 10-\lambda \end{vmatrix} = -\lambda((2-\lambda)(10-\lambda) - 24) = -\lambda(\lambda^2 - 12x - 4) = 0$$

$\lambda^2 - 12x - 4 = 0$

$$\frac{12 \pm \sqrt{144+16}}{2} = \frac{12 \pm \sqrt{160}}{2} = \frac{12 \pm 4\sqrt{10}}{2} = 6 \pm 2\sqrt{10}$$

2, $A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ $\quad b = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$

- Az $S(1)$ módszert! Konvergens-e a módszer
- Adjuk elvben hány lépés kell ... $\underline{x}^{0} = 0$ vektor ... 10^{-3}-os pontossághoz.

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 0 & -1 & 4 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & \frac{1}{4} & 1 & 0 \\ 0 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & \frac{1}{4} & 1 & 0 \\ 0 & 0 & 4 & \frac{1}{16} & \frac{1}{4} & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{16} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{64} & \frac{1}{16} & \frac{1}{4} \end{array}\right)$$

$(L+D)^{-1}$

$$\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{16} & \frac{1}{4} & 0 \\ \frac{1}{64} & \frac{1}{16} & \frac{1}{4} \end{pmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{16} & \frac{1}{4} \\ 0 & \frac{1}{64} & \frac{1}{16} \end{pmatrix}}_{B_S}$$

ⓐ

$$\|B_S\|_\infty = \frac{1}{16} + \frac{1}{4} = \frac{5}{16} < 1$$

$$\|B_S\|_1 = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} = \frac{21}{64}$$

$$\|B_S\|_F = \left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{16}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{64}\right)^2 + \left(\frac{1}{16}\right)^2\right)^{1/2} = \frac{1}{2^6}\left(2^9 + 2^5 + 1\right)^{1/2} \approx 0{,}365$$

HF
$$\varrho_{B_S} = \frac{1}{8}$$

$$\|\underline{x}^{(k)} - \underline{x}^*\|_\infty \le \frac{q^k}{1-q} \|\underline{x}^{(1)} - \underline{x}^{(0)}\|_\infty$$

$$\underline{x} = B_S \underbrace{\underline{x}^{(0)}}_{0} + c_S = c_S = (D+L)^{-1} b = \begin{bmatrix} 1/4 & 0 & 0 \\ 1/16 & 1/4 & 0 \\ 1/64 & 1/16 & 1/4 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/4 \\ 2/16 + \frac{6}{4} \\ 2/64 + 6/16 + 2/4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 13/8 \\ 29/32 \end{bmatrix} = \begin{matrix} 16/32 \\ 52/32 \\ 29/32 \end{matrix}$$

$$\|\underline{x}^{(k)} - \underline{x}^*\| \le \frac{(5/16)^k}{11/16} \|\underline{x}^{(1)} - \underline{x}^{(0)}\|_\infty < 10^{-3}$$

$$16/11 \cdot \left(\frac{5}{16}\right)^k \|\underline{x}^{(1)}\|_\infty < 10^{-3}$$

$$\frac{2\cancel{16}}{11}\left(\frac{5}{16}\right)^k \cdot \frac{13}{8} < 10^{-3}$$

$$\frac{26}{11}\left(\frac{5}{16}\right)^k < 10^{-3}$$

$$\left(\frac{5}{16}\right)^k < \frac{11}{26} 10^{-3}$$

$$\lg\left(\frac{5}{16}\right)^k < \lg\left(\frac{11}{26} \cdot 10^{-3}\right)$$

$$k \lg\left(\frac{5}{16}\right) < \lg\left(\frac{11}{26} 10^{-3}\right)$$

$$k > \frac{\lg \frac{11}{26} 10^{-3}}{\lg \frac{5}{16}} \approx 6{,}67$$

Előző

(1) $4x_1 - 1x_2 + 0x_3 = 2 \Rightarrow \cancel{x_1 = 2 + x_2} \Rightarrow x = \frac{1}{4}(2 + x_2)$

$-1x_1 - 4x_2 - 1x_3 = 6 \Rightarrow x_2 = \frac{1}{4}(6 + x_1 + x_3)$

$0x_1 - 1x_2 + 4x_3 = 2 \Rightarrow x_3 \frac{1}{4}(2 + x_2)$

$x_1^{(k)} = \frac{1}{4}(x_2^{(k-1)} + 2)$

$x_2^{k} = \frac{1}{4}(6 + x_1^{(k)} + x_3^{(k-1)})$

$x_3^{(k)} = \frac{1}{4}(2 + x_2^{(k)}$

$x_1^{(0)} = 0$

$x_2^{0} = 0$

$x_3^{0} = 0$

$x_1^{(1)} = \frac{1}{4}(x_2^{(0)} + 2) = \frac{1}{2}$

$x_2^{(1)} = \frac{1}{4}(6 + x_1^{(1)} + x_3^{(0)}) = \frac{1}{4}(6 + \frac{1}{2} + 0) = \frac{13}{8}$

$x_3^{(1)} = \frac{1}{4}(2 + x_2^{(1)}) = \frac{1}{4}(2 + \frac{13}{8}) = \frac{1}{4}(\frac{29}{8}) = \frac{29}{32}$

HF $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ $b = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}$

3 $A = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$ Melyen „a" értékére lesz konvergens J(1) S(1) iteráció

$B_J = -D^{-1}(L+U) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & a \\ a & a \end{bmatrix} = \begin{bmatrix} 0 & -a \\ -a & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$k_{B_J}(\lambda) = \begin{vmatrix} -\lambda & -a \\ -a & -\lambda \end{vmatrix} = \lambda^2 - a^2 = (\lambda - a)(\lambda + a) = 0$

$\lambda_1 = a$

$\lambda_2 = -a$

$\rho(B_J) = |a| < 1$ $\quad \boxed{-1 < a < 1}$

Ezen $\forall x_0$ kezdővektor esetén a J(1) iteráció konvergens.

$B_S = -(L+D)^{-1} \cdot U$

$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ a & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{a} & 0 \end{array}\right]$ $\quad \underbrace{}_{(L+D)^{-1}}$

$k_{B_S}(\lambda) = \begin{vmatrix} -\lambda & -a \\ 0 & 1-\lambda \end{vmatrix} = \lambda(1 - \lambda) = 0$

$\lambda_1 = 0$

$\lambda_2 = 1$

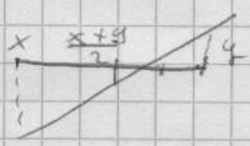
$\rho(B_S) = 1$

semilyen „a"-ra nem konv.

$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ -1/a & -1 \end{bmatrix} \begin{bmatrix} 0 & -a \\ 0 & +1 \end{bmatrix}$

1) $P(x) = x^3 + x^2 + 3x + 2$ Polinom egyetlen gyökét határozzuk intervallum felezéssel $1/10$-es pontossággal



$f(x) \cdot f(y) < 0$

$$\left.\begin{array}{l} P(0) = 2 > 0 \\ P(-1) = -1 < 0 \end{array}\right\} \Rightarrow [-1, 0]\text{-ban van gyök}$$

$\overset{-}{x_0} = -1 \qquad \overset{+}{y_0} = 0 \qquad y_0 - x_0 = 0 + 1 = 1$

$$|x_n - x^*| \le \frac{|y_0 - x_0|}{2^n} < \frac{1}{10}$$

$10 < 2^n$ 4 felezés elég

$\overset{-}{x_0} = -1 \qquad \overset{+}{y_0} = 0 \Rightarrow z_0 = \frac{x_0 + y_0}{2} = -\frac{1}{2}$

$P(z_0) = -\frac{1}{8} + \frac{1}{4} - \frac{3}{2} + 2 = \frac{5}{8} > 0$

$x_1 = -1 = \frac{2}{2} \quad y_1 = z_0 = -\frac{1}{2} \Rightarrow z_1 = \frac{x_1 + y_1}{2} = -\frac{3}{4}$

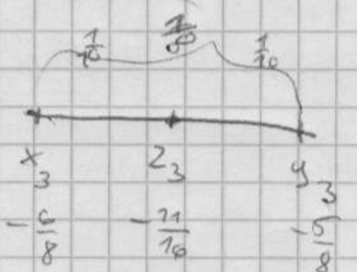
$P(-\frac{3}{4}) = -\frac{27}{64} + \frac{9}{16} - \frac{9}{4} + 2 = -\frac{7}{64} <$

$x_2 = z_1 = -\frac{3}{4} \quad y_2 = -\frac{1}{2} = -\frac{2}{4} \Rightarrow z_2 = -\frac{5}{8}$

$P(z_2) = -\frac{125}{8^3} + \frac{25}{8^2} - \frac{15}{8} + 2 = \frac{139}{512} > 0$

$x_3 = -\frac{3}{4} = -\frac{6}{8} \quad y_3 = -\frac{5}{8}$

$\boxed{z_3 = -\frac{11}{16}}$



3. Az $5x^3-20x+3=0$ $\quad x\in[0,1]$ $\quad M_0$-ra írjuk fel egy egyszerű iterációt

Becsüljük a hibát!

Hány lépést kell végezni hogy a hiba $< 10^{-2}$ ha $x_0=0$

$x=f(x)$ $\qquad f$ kontrakció

$|f(x)-f(y)| < q\,|x-y| \quad \forall x,y$

$q<1$

a) $x=\dfrac{5x^3+3}{20}$

b) $5x^3=20x-3$

$x^3=\dfrac{20x-3}{5}$

$x=\sqrt[3]{\dfrac{20x-3}{5}}$

c) $5x^2-19x+3=x$

a) $f(x)=\dfrac{5x^3+3}{20}=\dfrac{5}{20}x^3+\dfrac{3}{20}$

$\underset{\text{ÉT}}{D}f=[0;1] \Rightarrow \underset{\text{ÉK}}{R}_e\,[f(0),f(1)]$

mert f mig. nov.

$f'=\frac{15}{20}x^2>0$

$x^3-y^3=(x-y)(x^2+xy+y^2)$

$|f(x)-f(y)|=\left|\dfrac{5x^3+3}{20}-\dfrac{5y^3+3}{20}\right|=\left|\dfrac{5}{20}(x^3-y^3)\right|=\dfrac{5}{20}\cdot|x^3-y^3|=\dfrac{1}{4}|x-y|\cdot|\underbrace{x^2+xy+y^2}_{\le 3}|\le$

$0<x^2<1$

$0<xy<1$

$0<y^2<1$

$\le\dfrac{1}{4}|x-y|\,3=\dfrac{3}{4}\cdot|x-y|$

$q=\dfrac{3}{4}<1$

$\Rightarrow \exists!\ x^*\in[a,b] \quad f(x^*)=x^*$

$x_{r+1}=f(x_r)$ bármely x_0 esetén konvergens

$|x_r-x^*|\le\dfrac{q^r}{1-q}\cdot|x_1-x_0|$

$|x_r-x^*|\le\dfrac{\left(\frac{3}{4}\right)^r}{\frac{1}{4}}|x_1-x_0|=4\cdot\left(\dfrac{3}{4}\right)^r\cdot(x_1-x_0)$

$x_1=f(x_0)=\dfrac{5x_0^3+3}{20}=\dfrac{3}{20}$

$|x_r-x^*|\le 4\cdot\left(\dfrac{3}{4}\right)^r\left|\dfrac{3}{20}-0\right|<10^{-2}$

$\dfrac{3}{5}\cdot\left(\dfrac{3}{4}\right)^r<\dfrac{1}{100}$

$\lg\frac{3}{5}+\lg\left(\frac{3}{4}\right)^r<\lg\frac{1}{100}=-2$

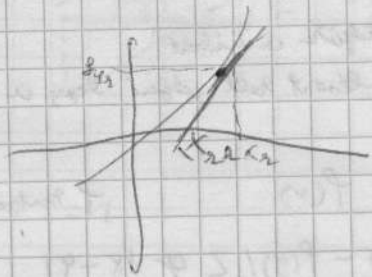
$r\underbrace{\lg\frac{3}{4}}_{<0}<-2-\lg\frac{3}{5}$

$r>\dfrac{-2-\lg\frac{3}{5}}{\lg\frac{3}{4}}\approx 14{,}83$

$r=15$ lépés elég

3) Írjuk fel az $f(x) = e^x + x = 0 \quad x \in \mathbb{R}$ no-ra a Newton módszert

Milyen x_0-ra konvergens az eljárás



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} + x_n}{e^{x_n} + 1} =$$

$$= \frac{e^{x_n} x_n + x_n - e^{x_n} - x_n}{e^{x_n} + 1} = \frac{e^{x_n}(x_n - 1)}{e^{x_n} + 1}$$

$$\left.\begin{array}{l} f(0) = 1 + 0 = 1 > 0 \\ f(-1) = \frac{1}{e} - 1 < 0 \end{array}\right\} \Rightarrow [-1; 0]\text{-n } \exists x^* \; f(x^*) = 0$$

1, $\exists x^* : f(x^*) = 0 \quad x^* \in [a, b]$ ✓

2, f' és f'' állandó előjelű

3, $x_0 \quad f(x_0) \cdot f''(x_0) > 0$

$f'(x) = e^x + 1 \geq 1 > 0 \quad \forall x \in \mathbb{R}$ ✓

$f''(x) = e^x > 0 \quad \forall x \in \mathbb{R}$ ✓

$f(x_0) \underbrace{f''(x_0)}_{>0} > 0 \iff f(x_0) > 0$

$f(x) = e^x + x \quad (f'(x) = e^x + 1 > 0)$

szig. mon. növ. $\quad x^* < x_0$ mindig jó

$x_0 = 0$ jó

↑

Monoton konvergencia

Lokális konvergencia

↓

1, $\exists x^* \in [a, b] \quad f(x^*) = 0$ ✓

2, $f'(x) \neq 0 \quad \forall x \in [a, b]$ ✓

3, $m_1 = \inf_{x \in [a,b]} |f'(x)| > 0$

$m_1 = 1 > 0$

4, $M_2 = \sup_{x \in [a,b]} |f''(x)| < \infty$

$= e^0 = 1$

5. $x_0 \quad r = \min\left\{\frac{2m_1}{M_2} ; |x_0 - a| ; |x_0 - b|\right\}$

$r = \min\{2, |x_0 + 1|, |x_0|\}$

4, Adjunk 10^{-1} pontosságú becslést $\sqrt[3]{2}$ értékére

$P(x)$ $\quad P(\sqrt[3]{2})=0$

$$x^3=2$$

$$P(x)=x^3-2=0$$

$$\boxed{\begin{array}{c} \cdot \sqrt[r]{2} \\ x^r=2 \\ P(x)=x^r-2 \end{array}}$$

$$1^3=1<2<8=2^3$$

$$1<\sqrt[3]{2}<2$$

$x_0=1 \qquad y_0=2$

$$\left.\begin{array}{l} P(1)=-1<0 \\ P(2)=6>0 \end{array}\right\} \Rightarrow [1,2]\text{-n van gyök}$$

$$|x^*-x^*|<\frac{y_0-x_0}{2^n}=\frac{1}{2^n}<10^{-1}$$

$$10<2^n$$

$$4\leq n$$

$x_0^-=1 \qquad x_0^+=2 \Rightarrow z_0=\frac{3}{2} \quad P(z_0)=\frac{27}{8}-2>0$

$x_1=1=\frac{2}{2} \qquad y_1=\frac{3}{2} \Rightarrow z_1=\frac{5}{4} \quad P(z_1)=\frac{125}{64}-2<0$

$x_2=\frac{5}{4} \qquad y_2=\frac{3}{2}=\frac{6}{4} \Rightarrow z_2=\frac{11}{8} \quad P(z_2)=\frac{307}{512}>0$

$x_3=\frac{5}{4}=\frac{10}{8} \qquad y_3=\frac{11}{8} \Rightarrow \boxed{z_3=\frac{21}{16}} \qquad \sqrt[3]{2}\approx\frac{21}{16}$