2007.02.22. csütörtök — II. Gyakorlat (2. hét)

(1) Elsőrendű lineáris diff.egyenletek

$$y'(x) + a(x)y(x) = h(x)$$; ha $h(x)=0 \Rightarrow$ homogén DE

H: $y'(x) + a(x)\,y(x) = 0 \Rightarrow$ szétválasztható változójú DE

$$\Leftrightarrow \int \frac{dy}{y} = -\int a(x)\,dx \qquad y \neq 0$$

$$\ln(y) = -\int a(x)\,dx + \ln c$$

ált. megold. (konst. faktorban) $y_H = c\,e^{-\int a(x)\,dx}$

IH: (partikuláris vált. m.o.) $\Rightarrow$ partikuláris m.o. y_{ip}

állandó variálásának módszere:

$$y_p = c(x)\,e^{-\int a(x)\,dx}$$

próbafv. — behelyettesítés

szétválasztható változójú egyenl. $c(x)$-re — $\boxed{y_{ip}}$

$$\Rightarrow \boxed{y_{ált} = y_H + y_{ip}}$$

példa (1)

$$y' - \frac{x}{x^2+4}y = 6x \qquad y(0)=4$$

H: $y' = \dfrac{x}{x^2+4}y$ $\quad \boxed{\alpha: y=0}$

$$\int \frac{dy}{y} = \int \frac{x}{x^2+4}\,dx$$

$$\ln|c| = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx = \frac{1}{2}\ln|x^2+4| + \ln c$$

$$\int \frac{f'}{f} = \ln|f| + c$$

$$\boxed{y_H = c\sqrt{x^2+4}}$$

IH

próbafv

$$y_p = c(x)\sqrt{x^2+4}$$

$$y_p' = c'\cdot\sqrt{x^2+4} + c\cdot\frac{1}{2}\frac{1}{\sqrt{x^2+4}}\cdot 2x$$

$$c'\cdot\sqrt{x^2+4} + \cancel{c\frac{x}{\sqrt{x^2+4}} - \frac{x}{x^2+4}c\sqrt{x^2+4}} = 6x$$

c-s tag ∀ kiesik !!!

$$\int dc = \int \frac{6x}{\sqrt{x^2+4}}\,dx \qquad \int f'\cdot f^{\alpha} = \frac{f^{\alpha+1}}{\alpha+1} \qquad \alpha \neq -1$$

$$c(x) = 3\int 2x\,(x^2+4)^{-1/2}\,dx = 3\,\frac{(x^2+4)^{1/2}}{1/2} = 6\sqrt{x^2+4}$$

$$\boxed{y_{ip} = 6\sqrt{x^2+4}\cdot\sqrt{x^2+4} = 6(x^2+4)}$$

$$\boxed{y_{ált} = y_H + y_{ip} = c\sqrt{x^2+4} + 6(x^2+4)}$$

$x=0$
$y=4$

$$4 = c\cdot 2 + 6\cdot 4$$

$$-10 = c$$

$$y = -10\sqrt{x^2+4} + 6(x^2+4)$$

példa (2)

$$y' - \frac{2}{x}y = x \qquad x \neq 0$$

H: $y' = \frac{2}{x}y$ $\quad \alpha)\ y = 0$

$\beta)$ ha $y \neq 0$ $\displaystyle\int \frac{dy}{y} = \int \frac{2}{x}\,dx$

$$\ln|y| = 2\ln|x| + \ln c$$

$$\boxed{y_H = c x^2}$$

IH: próbafv

$$y_p = c(x) \cdot x^2$$

$$y_p' = c' \cdot x^2 + c \cdot 2x$$

c-s kiesik $\quad c' x^2 + \cancel{2cx} - \cancel{\frac{2}{x} c x^2} = x$

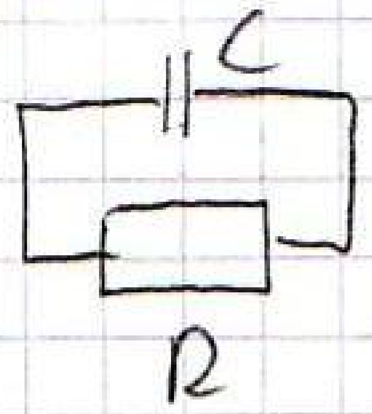
$$\int dc = \int \frac{1}{x}\,dx$$

$$c(x) = \ln|x|$$

$$\boxed{y_{ip} = \ln|x| \cdot x^2}$$

$$y_{ált} = cx^2 + \ln|x|\,x^2$$

<u>példa (3)</u> Kondenzátor kisülése



$t=0$ Q_c feltöltve a kondi

$Q(t)=?$

$$C=\frac{Q_c(t)}{U_c(t)} \quad \forall t$$

$$R=\frac{U_R(t)}{I(t)} \qquad I(t)=\dot{Q}(t)=\frac{dQ}{dt}$$

$$U_c+U_R=0$$

$$\frac{Q(t)}{C}+R\frac{dQ}{dt}=0 \qquad \text{(homogén : nincs benne fenger)}$$

$$\frac{dQ}{dt}+\frac{Q(t)}{RC}=0 \Rightarrow \int\frac{dQ}{Q}=\int-\frac{dt}{RC}$$

$$\ln|Q|=-\frac{t}{RC}+\ln k$$

$$\underline{Q(t)=k\cdot e^{-t/RC}}$$

$$Q(0)=Q_0$$

$$Q_0=k\left(e^{0}\right)$$

$$\underline{\underline{Q(t)=Q_0 e^{-t/RC}}}$$

<u>Egzakt diff.egyenletek</u>

(termodinamika + potenciál előadás)
$\Rightarrow$ teljes ínthető ke. el.

$$p(x,y)\,dx+q(x,y)\,dy=0$$

<u>egzakt</u>, ha $\boxed{\frac{\partial p}{\partial y}=\frac{\partial q}{\partial x}}$

megoldás:

$$F(x,y)=c \qquad \text{(potenciál)}$$

$$\frac{\partial F}{\partial x}=p(x,y)$$

$$\frac{\partial F}{\partial y}=q(x,y)$$

példa(1)

$$2x^3 - xy^2 + (2y^3 - x^2y)y' = 0$$

$$\underbrace{[2x^3 - xy^2]dx}_{p(x,y)} + \underbrace{[2y^3 - x^2y]dy}_{q(x,y)} = 0$$

$$\frac{\partial p}{\partial y} = -2xy$$

$$\frac{\partial q}{\partial y} = -2xy$$

$\Rightarrow\ = \Rightarrow$ egzakt

$\exists\ F(x,y)$ fv, hogy $\frac{\partial F}{\partial x} = 2x^3 - xy^2 \Rightarrow F(x,y) = \int 2x^3 - xy^2\,dx = \frac{2x^4}{4} - \frac{x^2y^2}{2} + c(y)$

$$\frac{\partial F}{\partial y} = 2y^3 - x^2y$$

$$\frac{\partial}{\partial y}$$

$$\frac{\partial F}{\partial y} = -x^2y + \frac{\partial c}{\partial y} = 2y^3 - x^2y$$

$$\frac{\partial c}{\partial y} = 2y^3$$

$$c(y) = \int 2y^3\,dy = 2\frac{y^4}{4} = \frac{y^4}{2}$$

$$F(x,y) = \frac{2x^4}{4} - \frac{x^2y^2}{2} + \frac{y^4}{2} = C$$

példa(2)

$$2xy + (1+x^2)y' = 0$$

$$\underbrace{2xy\,dx}_{p(x,y)} + \underbrace{(1+x^2)\,dy}_{q(x,y)} = 0$$

$$\frac{\partial p}{\partial y} = 2x$$

$$\Rightarrow = \Rightarrow \text{ egzakt } \Rightarrow$$

$$\frac{\partial q}{\partial y} = 2x$$

$$\Rightarrow \exists\, F(x,y)$$

$$\frac{\partial F}{\partial x} = 2xy \Rightarrow \boxed{F(x,y) = \int 2xy\,dx = x^2y + c(y)}$$

$$\downarrow \frac{\partial}{\partial y}$$

$$\frac{\partial F}{\partial y} = 1+x^2$$

$$\frac{\partial F}{\partial y} = x^2 + \frac{\partial c}{\partial y} = 1+x^2$$

$$\frac{\partial c}{\partial y} = 1$$

$$\boxed{c(y) = \int c\,dy = \int 1\,dy = y}$$

$$\underline{\underline{F(x,y) = x^2y + y = C}}$$

Egzakttá tehető DE-k Euler multiplikátorral

$$p(x,y)\,dx + q(x,y)\,dy = 0$$

$$\frac{\partial p}{\partial y} \neq \frac{\partial q}{\partial x} \Rightarrow \text{nem egzakt}$$

$/ \cdot \mu(x,y)$

$$\mu(x,y)\cdot\left[p(x,y)\,dx + q(x,y)\,dy\right] = 0$$

$$\frac{\partial \mu\cdot p}{\partial y} = \frac{\partial \mu q}{\partial x} \Rightarrow \text{egzakt} \Rightarrow \text{megold}: F(x,y)$$

$\boxed{\text{példa (1)}}$

$$(1-xy)\,dx + (xy - x^2)\,dy = 0$$

$$p(x,y) \qquad q(x,y)$$

$$\frac{\partial p}{\partial y} = -x$$

$\nparallel \Rightarrow$ nem egzakt

$$\frac{\partial q}{\partial x} = y - 2x$$

T.f.h. $\exists$ $\mu(x)$ integráló tényező

$$\frac{\partial[\mu(x)(1-xy)]}{\partial y} = \frac{\partial[\mu(x)(xy-x^2)]}{\partial x}$$

$$\mu(x)(-x) = \frac{\partial \mu}{\partial x}(xy-x^2) + \mu(x)(y-2x)$$

$$\mu(-x+2x-y) = \frac{d\mu}{dx}(xy-x^2)$$

$$\frac{x-y}{x(y-x)}\,dx = \frac{d\mu}{\mu}$$

$$\int \frac{d\mu}{\mu} = -\int \frac{1}{x}\,dx$$

$$\ln|\mu| = -\ln|x|$$

$$\boxed{\mu(x) = \frac{1}{x}}$$

$$\left(\frac{1}{x} - y\right)dx + (y-x)dy = 0 \quad \text{egzakt}$$

$$\Rightarrow \exists\ F(x,y)$$

$$\frac{\partial F}{\partial x} = \frac{1}{x} - y \Rightarrow F(x,y) = \int\left(\frac{1}{x} - y\right)dx = \ln|x| - xy + c(y)$$

$$\Big\downarrow \frac{\partial}{\partial y}$$

$$\frac{\partial F}{\partial y} = y - x$$

$$\frac{\partial F}{\partial y} = 0 - x + \frac{\partial c}{\partial y} = y - x \Rightarrow \frac{\partial c}{\partial y} = y$$

$$c(y) = \int y\,dy = \frac{y^2}{2}$$

$$F(x,y) = \ln|x| - xy + \frac{y^2}{2} = C$$

((eredetinek is megold)

példa(2)

$$\underbrace{y}_{p(x,y)}dx \underbrace{-(x+y)}_{q(x,y)}dy = 0$$

$$\frac{\partial p}{\partial y} = 1$$

$$\neq \Rightarrow \text{nem egzakt}$$

$$\frac{\partial q}{\partial x} = -1$$

tfh: $\exists\ \mu(x) = \mu$

$$\frac{\partial(\mu(x)y)}{\partial y} = \frac{\partial\left(\mu(x)[-x-y]\right)}{\partial x}$$

$$\mu(x) = \frac{\partial\mu}{\partial x}(-x-y) + \mu(x)(-1)$$

$$2\mu(x) = \frac{\partial\mu}{\partial x}[-x-y]$$

$$\frac{d\mu}{\mu} = -\frac{2}{x+y}\,dx$$ ↯

függ y-tól ⇒ ∄ csak x-től függő μ

tfh $\exists\ \mu = \mu(y)$

$$\frac{\partial[\mu(y)y]}{\partial y} = \frac{\partial[\mu(y)(-x-y)]}{\partial x}$$

$$\frac{d\mu}{dy}y + \mu(y) = -\mu(y)$$

$$\frac{d\mu}{dy} = -2\mu(y)$$

$$\int\frac{d\mu}{\mu} = -\int 2\,dy$$

$$\ln|\mu| = -2\ln|y|$$

$$\boxed{\mu(y) = \frac{1}{y^2}} \Rightarrow \text{HF}$$