2007.03.08. csütörtök IV. Gyakorlat (4. hét)

Magasabbrendű lin. áll. együtthatós IH DE

$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = h(x)$

H: $\Rightarrow y_H$ (tartalmaz n db konstanst)

próba függvény módszer: $h(x) = e^{\alpha x}(P_1(x)\cos\beta x + P_2(x)\sin\beta x)$

P_1, P_2 legfeljebb k-adfokú polinomok

két eset

① $\lambda = \alpha \pm i\beta$ nem gyöke a karakterisztikus polinomnak ($\nexists$ külső rezonancia)

$y_p = e^{\alpha x}(Q_1(x)\cos\beta x + Q_2(x)\sin\beta x)$

Q_1, Q_2 általános k-adfokú polinom

② $\lambda = \alpha \pm i\beta$ m-szeres gyöke a karakterisztikus polinomnak

(m-szeres ~~külső~~ külső rezonancia)

$y_p = x^m e^{\alpha x}(Q_1(x)\cos\beta x + Q_2(x)\sin\beta x)$

példa (1)

$y'' - 5y' + 6y = 2xe^x$

H: $y'' - 5y' + 6y = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda-2)(\lambda-3) = 0$

$\lambda_1 = 2$ 1x-es gyök

$\lambda_2 = 3$ 1x-es gyök

($\nexists$ belső rez.)

$\boxed{y_H = c_1 e^{2x} + c_2 e^{3x}}$

IH: $\Rightarrow \left.\begin{matrix}\alpha = 1\\ \beta = 0\end{matrix}\right\} \lambda = 1$ ($\Rightarrow$ nincs külső rez)

$y_p = (Ax + B)e^x$

$y_p' = Ae^x + (Ax+B)e^x = (A + Ax + B)e^x$

$y_p'' = Ae^x + (Ax + A + B)e^x = (Ax + 2A + B)e^x$

$\cancel{e^x}(Ax + 2A + B - 5Ax - 5A - 5B + 6Ax + 6B) = 2x\cancel{e^x}$

$x(2A) - 3A + 2B = 2x$

$2A = 2 \Rightarrow A = 1$

$-3A + 2B = 0 \Rightarrow B = \frac{3}{2}$

$\boxed{y_{ip} = \left(x + \frac{3}{2}\right)e^x}$

$$\boxed{y_{ált} = y_H + y_{ip} = c_1 e^{2x} + c_2 e^{3x} + \left(x + \frac{3}{2}\right)e^x}$$

példa (2)

$$y''-y'-2y=3e^{2x} \qquad y(0)=3$$
$$y'(0)=1$$

H: $\lambda^2-\lambda-2=0$

$(\lambda-2)(\lambda+1)=0$

$\lambda_1=2$ 1x-es gyök (nincs belső rez)

$\lambda_2=-1$ 1x-es gyök

$$\boxed{y_H=c_1e^{2x}+c_2e^{-x}}$$

$$\boxed{y_{ip}=x\cdot e^{2x}}$$

IH: $\left.\begin{matrix}\alpha=2\\ \beta=0\end{matrix}\right\}\lambda=2 \Rightarrow$ 1x-es külső rezonancia

$y_p=Ae^{2x}\cdot x$

$y_p'=2Ae^{2x}\cdot x+Ae^{2x}=e^{2x}(2Ax+A)$

$y_p''=2e^{2x}(2Ax+A)+2e^{2x}=e^{2x}(4Ax+4A)$

$\cancel{e^{2x}}(\cancel{4Ax}+4A-\cancel{2Ax}-A-\cancel{2Ax})=3\cancel{e^{2x}}$

$3A=3 \Rightarrow A=1$

$y_{ált}=c_1e^{2x}+c_2e^{-x}+xe^{2x}$

$y'_{ált}=2c_1e^{2x}-c_2e^{-x}+e^{2x}+2xe^{2x}$

$y(0)=3 \Rightarrow 3=c_1+c_2$

$y'(0)=1 \Rightarrow 1=2c_1-c_2+1 \Rightarrow c_2=2c_1$

$3=c_1+2c_1=3c_1 \Rightarrow c_1=1$
$c_2=2$

$$\boxed{y=e^{2x}+2e^{-x}+xe^{2x}}$$

példa (3)

$$y'''-2y''-y'+2y=\operatorname{ch}2x=\frac{e^{2x}+e^{-2x}}{2}$$

H: $\lambda^3-2\lambda^2-\lambda+2=0$

$\lambda^2(\lambda-2)-(\lambda-2)=0$

$(\lambda-2)(\lambda^2-1)=0$

$(\lambda-2)(\lambda-1)(\lambda+1)=0$

$\left.\begin{matrix}\lambda_1=2\\ \lambda_2=1\\ \lambda_3=-1\end{matrix}\right\}$ 1x-es gyökök, $\nexists$ belső rez.

$$\boxed{y_H=c_1e^{2x}+c_2e^{x}+c_3e^{-x}}$$

IH: $\left.\begin{matrix}\alpha_1=2\\ \beta_1=0\end{matrix}\right\}\lambda_1=2 \quad \left.\begin{matrix}\alpha_2=-2\\ \beta_2=0\end{matrix}\right\}\lambda_2=-2$

1x-es külső rez $\qquad$ $\nexists$ külső rez

$y_p=Ae^{2x}\cdot x+Be^{-2x}$

$y_p'=2Ae^{2x}\cdot x+Ae^{2x}-2Be^{-2x}=$
$=e^{2x}(2Ax+A)-2Be^{-2x}$

$y_p''=2e^{2x}(2Ax+A)+2Ae^{2x}+4Be^{-2x}=$
$=e^{2x}(4Ax+4A)+4Be^{-2x}$

$y_p'''=2e^{2x}(4Ax+4A)+4Ae^{2x}-8Be^{-2x}=$
$=e^{2x}(8Ax+12A)-8Be^{-2x}$

megoldik egymást

$$e^{2x}\left(\cancel{8Ax}+12A-\cancel{8Ax}-8A-\cancel{2Ax}-A+\cancel{2Ax}\right)+$$
$$+\,e^{-2x}\left(-8B-8B+2B+2B\right)=\frac{e^{2x}}{2}+\frac{e^{-2x}}{2}$$

$$3A=\frac{1}{2}\Rightarrow A=\frac{1}{6}$$
$$-12B=\frac{1}{2}\Rightarrow B=-\frac{1}{24}$$

$$y_{ip}=\frac{1}{6}xe^{2x}-\frac{1}{24}e^{-2x}$$

Laplace-transzformáció:

$f:\mathbb{R}\mapsto\mathbb{C}$ $\quad f(t)\equiv 0\quad t<0$ bázisfüggvények mint $\mathcal{F}$: cos, sin

$t\in\mathbb{R}$ $\quad \mathcal{L}f: F(z)=\int_0^{\infty} e^{-zt} f(t)\,dt$

tulajd.: ① $\alpha f+\beta g\mapsto \alpha F+\beta G$ (lineáris)

② $f'(t)\mapsto z\cdot F(z)-f(0)$ (deriválási) ⇒ polinomiális rendel

$f''(t)\mapsto z^2F(z)-zf(0)-f'(0)$

$f^{n}(t)\mapsto z^nF_z-z^{(n-1)}f(0)-z^{n-2}f'(0)-z^{n-3}f''(0)-\dots-f^{(n-1)}(0)$

③ $(-1)^nt^nf(t)\mapsto F^{(n)}(z)$ (kitolási gerj)

④ $\int_0^t f(\tau)d\tau\mapsto\frac{F(z)}{z}$

⑤ $e^{z_0t}f(t)\mapsto F(z-z_0)$ (eltolási tétel)

⑥ $f(t-t_0)\mapsto e^{-t_0z}F(z)$ (késleltetés)

⑦ $\int_0^t f(\tau)g(t-\tau)d\tau\mapsto F(z)\cdot G(z)$ (konvolúciós)

$f(t)$	$\mapsto$ $F(z)$
1	$\frac{1}{z}$
$\frac{t^n}{n!}$	$\frac{1}{z^{n+1}}$
$e^{\alpha t}$	$\frac{1}{z-\alpha}$
$\cos\beta t$	$\frac{z}{z^2+\beta^2}$
$\sin\beta t$	$\frac{\beta}{z^2+\beta^2}$
$\mathrm{ch}\,\beta t$	$\frac{z}{z^2-\beta^2}$
$\mathrm{sh}\,\beta t$	$\frac{\beta}{z^2-\beta^2}$
$e^{\alpha t}\cos\beta t$	$\frac{z-\alpha}{(z-\alpha)^2+\beta^2}$
$e^{\alpha t}\sin\beta t$	$\frac{\beta}{(z-\alpha)^2+\beta^2}$
$\frac{t^n}{n!}e^{\alpha t}$	$\frac{1}{(z-\alpha)^{n+1}}$

példa(4) $\ddot{x}(t) = \dfrac{d^2x(t)}{dt^2}$

$$\ddot{x} + 4\dot{x} + 4x = t^3 e^{-2t} \qquad \begin{matrix} x(0)=1 \\ \dot{x}(0)=1 \end{matrix}$$

$x(t) \mapsto X(z)$

$\dot{x}(t) \mapsto z \cdot X(z) - x(0) = z X(z) - 1$

$\ddot{x}(t) \mapsto z^2 X(z) - z x(0) - x(0) = z^2 X(z) - z - 1$

$\begin{matrix} n=3 \\ \alpha=-2 \end{matrix}$ $t^3 e^{-2t} \mapsto 3! \dfrac{1}{(z+2)^4} = \dfrac{6}{(z+2)^4}$

$$z^2 X(z) - z - 1 + 4z X(z) - 4 + 4X(z) = \frac{6}{(z+2)^4}$$

$$X(z)\underbrace{(z^2+4z+4)}_{(z+2)^2} = \frac{6}{(z+2)^4} + z + 5$$

$\dfrac{6}{(z+2)^6} \mapsto 6 \cdot \dfrac{t^5}{5!} e^{-2t} = \dfrac{t^5}{20} e^{-2t}$ $\quad \begin{matrix} n=5 \\ \alpha=-2 \end{matrix}$

$$X(z) = \frac{6}{(z+2)^6} + \frac{z+5}{(z+2)^2}$$

$$\frac{z+5}{(z+2)^2} = \frac{z+2+3}{(z+2)^2} = \frac{1}{(z+2)} + \frac{3}{(z+2)^2}$$

$\dfrac{1}{(z+2)} \to e^{-2t}$, $\quad \dfrac{3}{(z+2)^2} \to 3te^{-3t}$

$$X(z) \mapsto x(t) = \frac{t^5}{20} e^{-2t} + e^{-2t} + 3te^{-2t}$$

példa(5)

$$y'' + y' - y = \cos t \qquad \begin{matrix} y(0)=0 \\ y'(0)=0 \end{matrix} \qquad \overset{HF}{\Rightarrow} y'' + 2y' + y = \cos t$$

$y(x) \mapsto Y(z)$

$y'(x) \mapsto z\,Y(z) - y(0) = z\,Y(z)$

$y''(x) \mapsto z^2 Y_z - z y(0) - y'(0) = z^2 Y(z)$

$\cos t \mapsto \dfrac{z}{z^2+1}$

$$z^2 Y(z) + z\,Y(z) - Y(z) = \frac{z}{z^2+1}$$

$$Y(z)\left[z^2 + z - 1\right] = \frac{z}{z^2+1}$$

$$D = 1 - 4(-1) = 5$$

$$z_{1,2} = \frac{-1 \pm \sqrt{5}}{2} \; \begin{matrix} \dfrac{-1+\sqrt{5}}{2} = \alpha_1 \\ \dfrac{-1-\sqrt{5}}{2} = \alpha_2 \end{matrix}$$

$$Y(z) = \frac{z}{(z^2+1)(z-\alpha_1)(z-\alpha_2)} = \frac{Az+B}{z^2+1} + \frac{C}{z-\alpha_1} + \frac{D}{z-\alpha_2}$$

Lin. áll. együtthatós DE-rendszerek

Homogén:

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$\dot{x}_2 = a_{21}x_1 + a_{12}x_2 + \dots + a_{2n}x_2$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{m}x_n$$

$$\underline{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{mn} \end{pmatrix} \quad \dot{\underline{x}} = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix}$$

$$\dot{\underline{x}} = \underline{\underline{A}} \cdot \underline{x}$$

$\hookrightarrow$ $\underline{\underline{A}}$ sajátértékei $\lambda_1, \lambda_2, \dots, \lambda_n$ 1x-es valós sajátértékek

sajátvektorok $\underline{s}_1, \underline{s}_2, \dots, \underline{s}_n$

$$\underline{x}(t) = c_1 e^{\lambda_1 t}\underline{s}_1 + c_2 e^{\lambda_2 t}\underline{s}_2 + \dots + c_n e^{\lambda_n t}\underline{s}_n \quad \text{ált. megold}$$

$$\underline{\underline{\Phi}} = \left(e^{\lambda_1 t}\underline{s}_1 \;\middle|\; e^{\lambda_2 t}\underline{s}_2 \;\middle|\; \dots \;\middle|\; e^{\lambda_n t}\underline{s}_n \right)$$

alapmátrix

példa (6)

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -3x_1 - 4x_2 \end{aligned}\right\}$$

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{E}}) = \det \begin{pmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{pmatrix} = -\lambda(-4-\lambda) + 3 = \lambda^2 + 4\lambda + 3 =$$

$$= (\lambda+3)(\lambda+1) = 0$$

$$\boxed{\lambda_1 = -3, \; \lambda_2 = -1}$$

$\lambda_1 = -3$ $\quad \underline{s}_1 = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$

$$\underline{\underline{A}}\,\underline{s} = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} s_2 \\ -3s_1 - 4s_2 \end{pmatrix} = \begin{pmatrix} -3s_1 \\ -3s_2 \end{pmatrix}$$

$$\begin{aligned} 3s_1 + s_2 &= 0 \\ -3s_1 - s_2 &= 0 \end{aligned} \quad \overset{\text{ha}}{\Rightarrow} \; s_1 = 1 \Rightarrow s_2 = -3 \qquad \underline{s}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$\boxed{\lambda_2 = -1}$ $\quad s_2 = -s_1 \quad \Rightarrow s_1 = 1 \Rightarrow s_2 = -1$

$-3s_1 - 4s_2 = -s_2 \qquad \underline{s}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\boxed{\begin{aligned} x_1(t) &= c_1 e^{-3t} + c_2 e^{-t} \\ x_2(t) &= -3c_1 e^{-3t} - c_2 e^{-t} \end{aligned}}$$