2007.03.22 csütörtök

Gyakorlat
V. ~~előadás~~ (6. hét)

Homogén állandó-együtthatós DE rendszer

$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}$ $\quad\underline{\underline{A}} \in (n \times n)$-es mátrix

$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \underline{\dot{x}} = \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix}$

① sajátvektorok segítségével

$\dot{x}_1 = x_2$

$\dot{x}_2 = -3x_1 - 4x_2$

$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}$

$$|\underline{\underline{A}} - \lambda \underline{\underline{E}}| = \begin{vmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = -\lambda(-4-\lambda) + 3 = \lambda^2 + 4\lambda + 3 = 0$$

$(\lambda+3)(\lambda+1) = 0$

$\lambda_1 = -3$

$\lambda_2 = -1$

Saját vektorok:

$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v} \qquad \underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\underline{\underline{A}}\,\underline{v} = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ -3v_1 - 4v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$\lambda_1 = -3$

$$\left.\begin{aligned} v_2 &= -3v_1 \\ -3v_1 - 4v_2 &= -3v_2 \end{aligned}\right\} \quad \begin{aligned} v_1 &= 1 \\ v_2 &= -3 \end{aligned}$$

$$\underline{v_1} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$\lambda_2 = -1$

$$\left.\begin{aligned} v_2 &= -1v_1 \\ -3v_1 - 4v_2 &= -v_2 \end{aligned}\right\} \quad \begin{aligned} v_1 &= 1 \\ v_2 &= -1 \end{aligned}$$

$$\Rightarrow \underline{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\underline{x} = c_1 e^{\lambda_1 t} \underline{v_1} + c_2 e^{\lambda_2 t} \underline{v_2}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \boxed{\begin{aligned} x_1 &= c_1 e^{-3t} + c_2 e^{-t} \\ x_2 &= -3c_1 e^{-3t} - c_2 e^{-t} \end{aligned}}$$

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$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} \qquad \underline{\underline{\Phi}}(t) = e^{\underline{\underline{A}}t} \qquad \underline{\underline{\Phi}}(t) = \left(\underline{x_1}, \underline{x_2}, \underline{x_n} \right)$$

alaprendszer

$$\underline{x} = c_1\underline{x_1} + c_2\underline{x_2} + \dots + c_n\underline{x_n}$$

$$\left(\underline{\underline{\dot{\Phi}}} t = \underline{\underline{A}} e^{\underline{\underline{A}}t} = \underline{\underline{A}}\Phi(t) \right)$$

Taylor:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{\underline{\underline{A}}t} = \sum_{k=0}^{\infty} \frac{t^k \underline{\underline{A}}^k}{k!}$$

∞ ? $\Rightarrow$ Lagrange

$\underline{\underline{A}} \in M_n \in n \times n$ -es matrix

$\forall$ sajátértéke egyszeres (nem degenerált)

$\exists$ $(n-1)$-edfokú polinom (Lagrange polinom)

$$\ell(\lambda, t) = \ell_{n-1}(t)\lambda^{n-1} + \ell_{n-2}(t)\lambda^{n-2} + \ldots + \ell_1(t)\lambda + \ell_0(t)$$

$$\ell(\lambda_k, t) = e^{\lambda_k t} \qquad k = 1, \ldots, n$$

$$\ell(\underline{\underline{A}}, t) = e^{\underline{\underline{A}}t}$$

$$\dot{x}_1 = 2x_1 + x_2$$
$$\dot{x}_2 = 3x_1 + 4x_2 \qquad \underline{\underline{A}} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

$$|\underline{\underline{A}} - \lambda \underline{\underline{E}}| = \begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda-1)(\lambda-5) = 0$$

$$\boxed{\lambda_1 = 1 \qquad \lambda_2 = 5} \Rightarrow \text{1x-es sajátért} \Rightarrow \text{Lagrange}$$

$$\ell(\lambda, t) = \ell_1 \lambda + \ell_0$$

$$\left.\begin{aligned} \ell(1,t) &= \ell_1 + \ell_0 = e^t \\ \ell(5,t) &= 5\ell_1 + \ell_0 = e^{5t} \end{aligned}\right\}$$

$$4\ell_1 = {}^{5t} - e^t \Rightarrow \boxed{\ell_1 = \frac{e^{5t} - e^t}{4}}$$

$$\boxed{\ell_0 = e^t - \ell_1 = e^t - \frac{e^{5t} - e^t}{4} = \frac{5e^t - e^{5t}}{4}}$$

$$e^{\underline{\underline{A}}t} = \frac{e^{5t} - e^t}{4}\underline{\underline{A}} + \frac{5e^t - e^{5t}}{4}\underline{\underline{E}} = \frac{e^{5t} - e^t}{4}\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} + \frac{5e^t - e^{5t}}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{e^{5t} + 3e^t}{4} & \frac{e^{5t} - e^t}{4} \\ \frac{3e^{5t} - 3e^t}{4} & \frac{3e^{5t} + e^t}{4} \end{pmatrix} = \underline{\underline{\phi}}(t)$$

$$x_1 = c_1 \frac{e^{5t} + 3e^t}{4} + c_2 \frac{e^{5t} - e^t}{4}$$

$$x_2 = c_1 \frac{3e^{5t} - 3e^t}{4} + c_2 \frac{3e^{5t} + e^t}{4}$$

③ Laplace-szal:

$$\left.\begin{aligned}\dot{x} &= x+y \\ \dot{y} &= x-y\end{aligned}\right\} \qquad \begin{aligned}x(0)&=0\\ y(0)&=1\end{aligned}$$

$x(t) \xrightarrow{\mathcal{L}} X(z)$ $\quad \dot{x}(t) \mapsto zX(z) - x(0) = zX$

$y(t) \xrightarrow{\mathcal{L}} Y(z)$ $\quad \dot{y}(t) \mapsto zY(z) - y(0) = zY - 1$

$$\left.\begin{aligned}zX &= X+Y \\ zY-1 &= X-Y\end{aligned}\right\} \Rightarrow \begin{aligned}&Y = X(z-1)\\ &zX(z-1)-1 = X - X(z-1)\end{aligned}$$

$X\left(z(z-1) - 1 + z - 1\right) = 1$

$X\left((z-1)(z+1) - 1\right) = 1$

$$\boxed{X = \frac{1}{z^2-2}}$$

$$\boxed{Y = \frac{z-1}{z^2-2}}$$

$\frac{\beta}{z^2-\beta^2} \xmapsto{\mathcal{L}^{-1}} \operatorname{sh}\beta t$

$\frac{1}{z^2-2} = \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{z^2-\sqrt{2}^2} \xmapsto{\mathcal{L}} \frac{1}{\sqrt{2}} \operatorname{sh}\sqrt{2}t = x(t)$

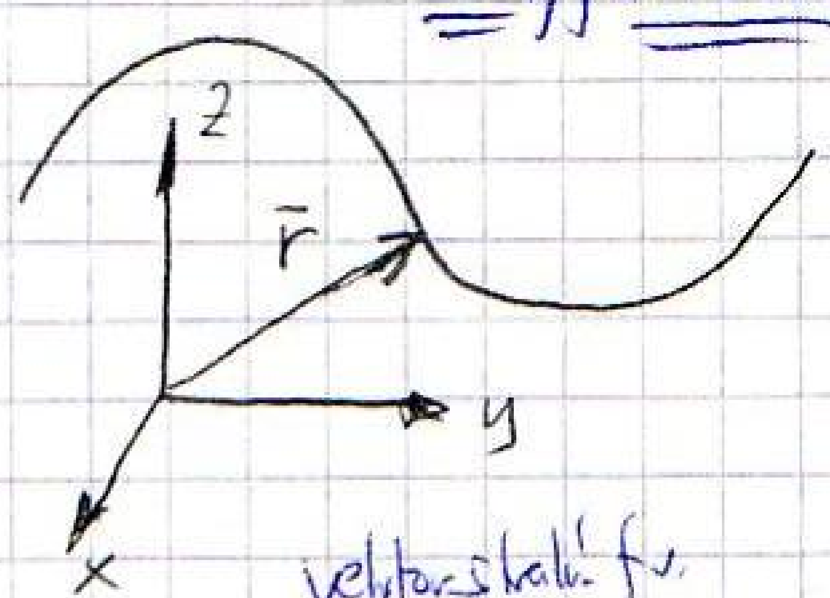
$\operatorname{sh}x = \frac{e^x - e^{-x}}{2}$

$\frac{z}{z^2-\beta^2} \xmapsto{\mathcal{L}^{-1}} \operatorname{ch}\beta t$

$\operatorname{ch}x = \frac{e^x + e^{-x}}{2}$

$\frac{z-1}{z^2-2} = \frac{z}{z^2-2} - \frac{1}{z^2-2} \mapsto \operatorname{ch}\sqrt{2}t - \frac{1}{\sqrt{2}}\operatorname{sh}\sqrt{2}t = y(t)$

Differenciál geometria

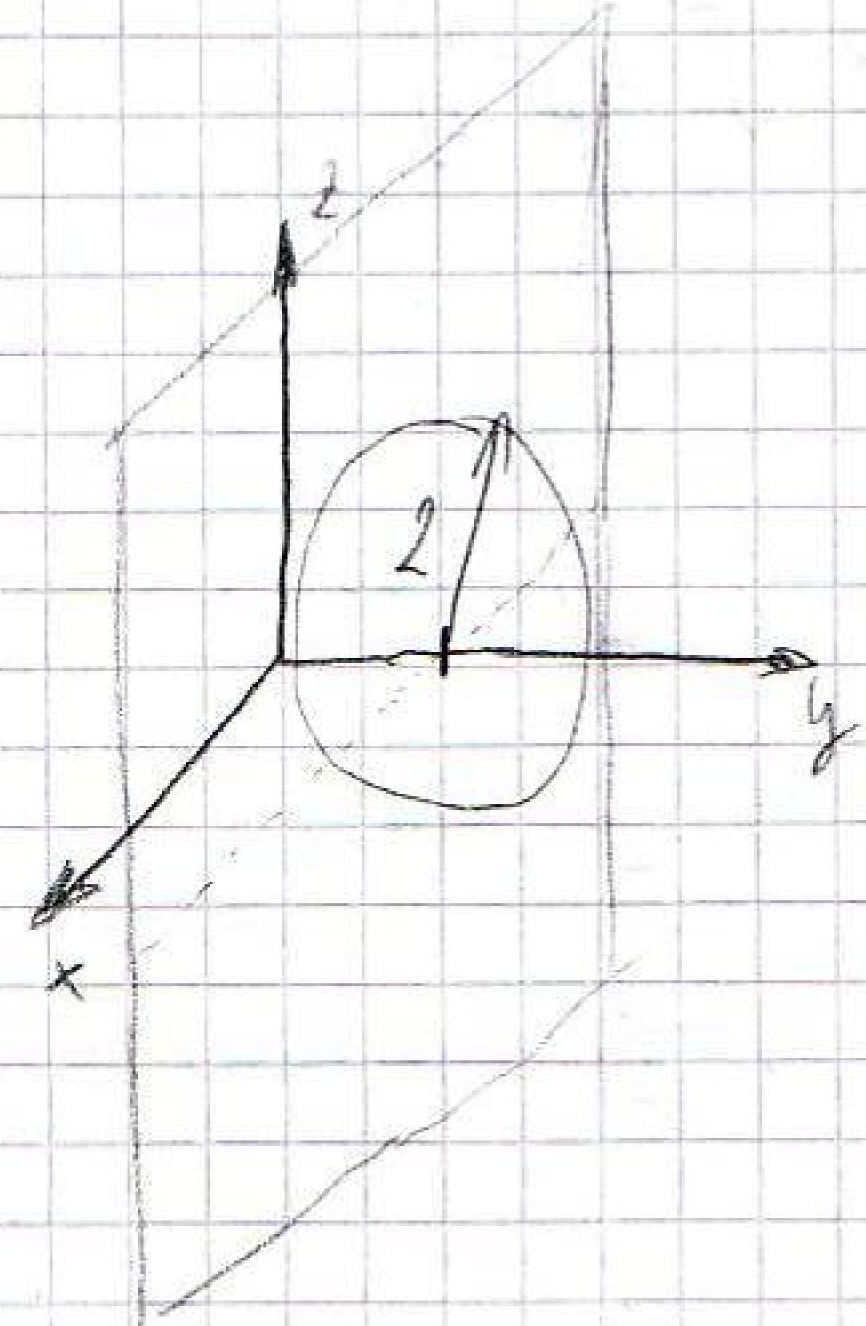


$\Rightarrow$ parametrizálás

vektorskalár fv.

$t \mapsto \underline{r}(t) = (x(t), y(t), z(t))$

a) $(0,1,0)$ kp-ú 2 sugarú kör
$\{y = 1, (x,y,z) \in \mathbb{R}^3\}$ síkban?



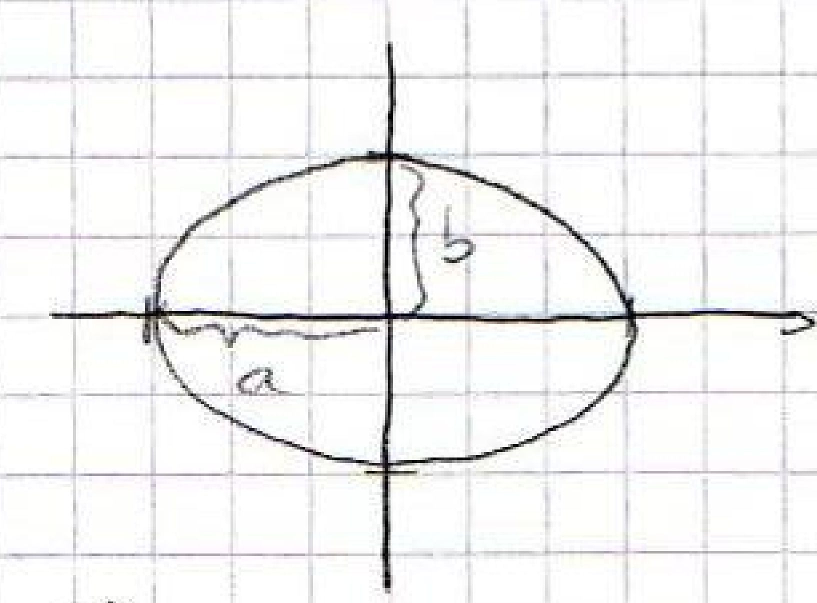
$\underline{r}(t) = (2\cos t, 1, 2\sin t)$

$y = 1$

$x = 2\cos t$

$z = 2\sin t$

b) $\{z = 0, (x,y,z) \in \mathbb{R}^3\}$ síkban
$(1,3)$ kp-ú a,b féltengelyű ellipszis



$x = a\cos t$, $\left(\frac{x}{a}\right)^2 = \cos^2 t$

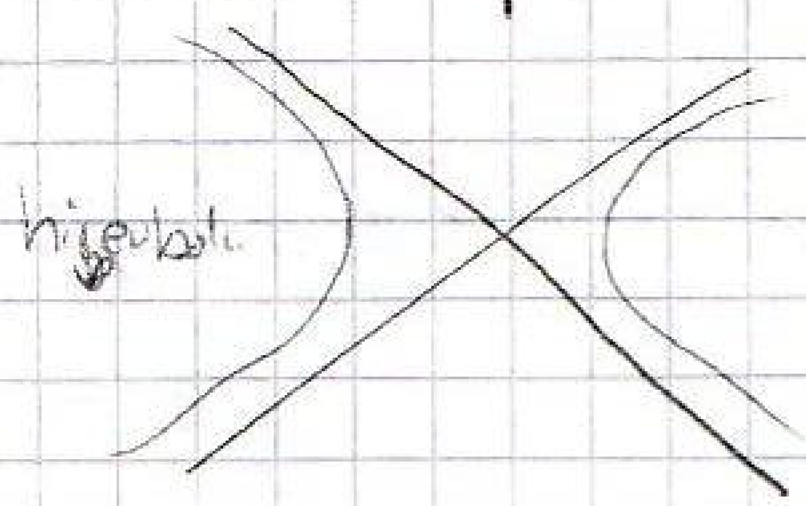
$y = b\sin t$, $\left(\frac{y}{b}\right)^2 = \sin^2 t$

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$

$x = x_0 + a\cos t$

$y = y_0 + b\sin t$



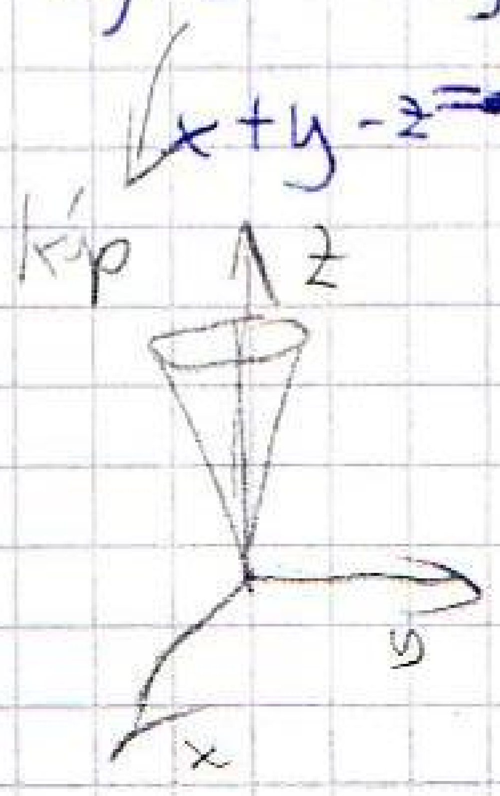
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$x = a\,\mathrm{ch}\,t$

$y = b\,\mathrm{sh}\,t$

$\underline{r}(t) = (1 + a\cos t, 3 + b\sin t, 0)$

c) $\left.\begin{array}{l} z = x^2 + y^2 \\ x + y - z = -4 \end{array}\right\}$ felületek metszésgörbéje



$\Downarrow$ $z = x + y + 4 \Rightarrow x + y + 4 = x^2 + y^2$

$$\underbrace{x^2-x}+\underbrace{y^2-y}=4$$

$$\left(x-\frac{1}{2}\right)^2-\frac{1}{4}+\left(y-\frac{1}{2}\right)^2-\frac{1}{4}=4$$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{9}{2} \qquad \text{k.p.} \left(\frac{1}{2};\frac{1}{2}\right)$$

$$r=\frac{3}{\sqrt{2}}$$

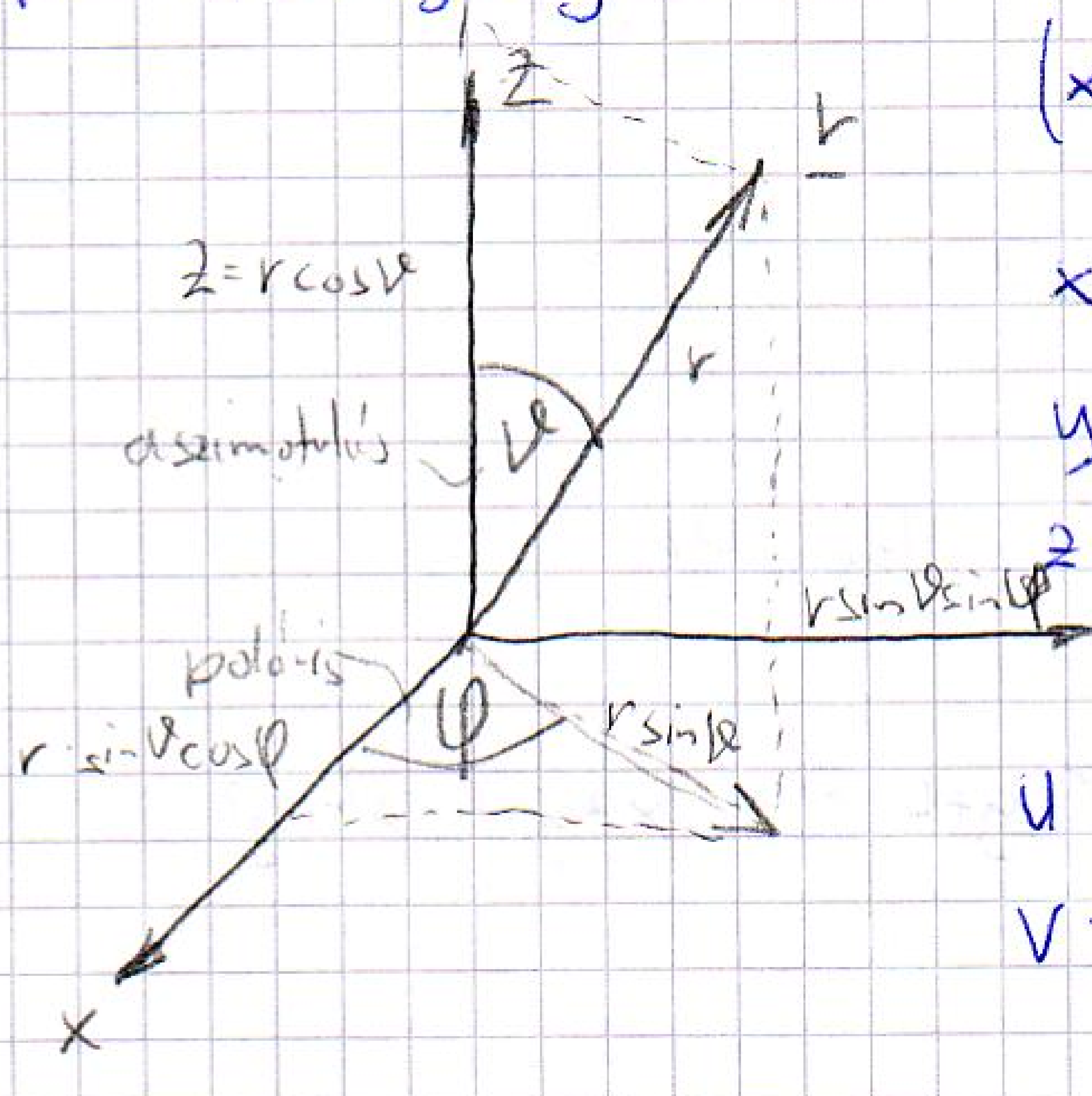
$$x=\frac{1}{2}+\frac{3}{\sqrt{2}}\cos t$$

$$y=\frac{1}{2}+\frac{3}{\sqrt{2}}\sin t$$

$$z=1+\frac{3}{\sqrt{2}}\left(\cos t+\sin t\right)+4$$

$r(u,v)$

(1,2,3) kp-ú $r=5$ sugarú gömb



$$(x,y,z)\Leftrightarrow(r,\vartheta,\varphi)$$

$$x=r\sin\vartheta\cos\varphi$$

$$y=r\sin\vartheta\sin\varphi$$

$$z=r\cos\vartheta$$

$$u\Rightarrow\vartheta$$

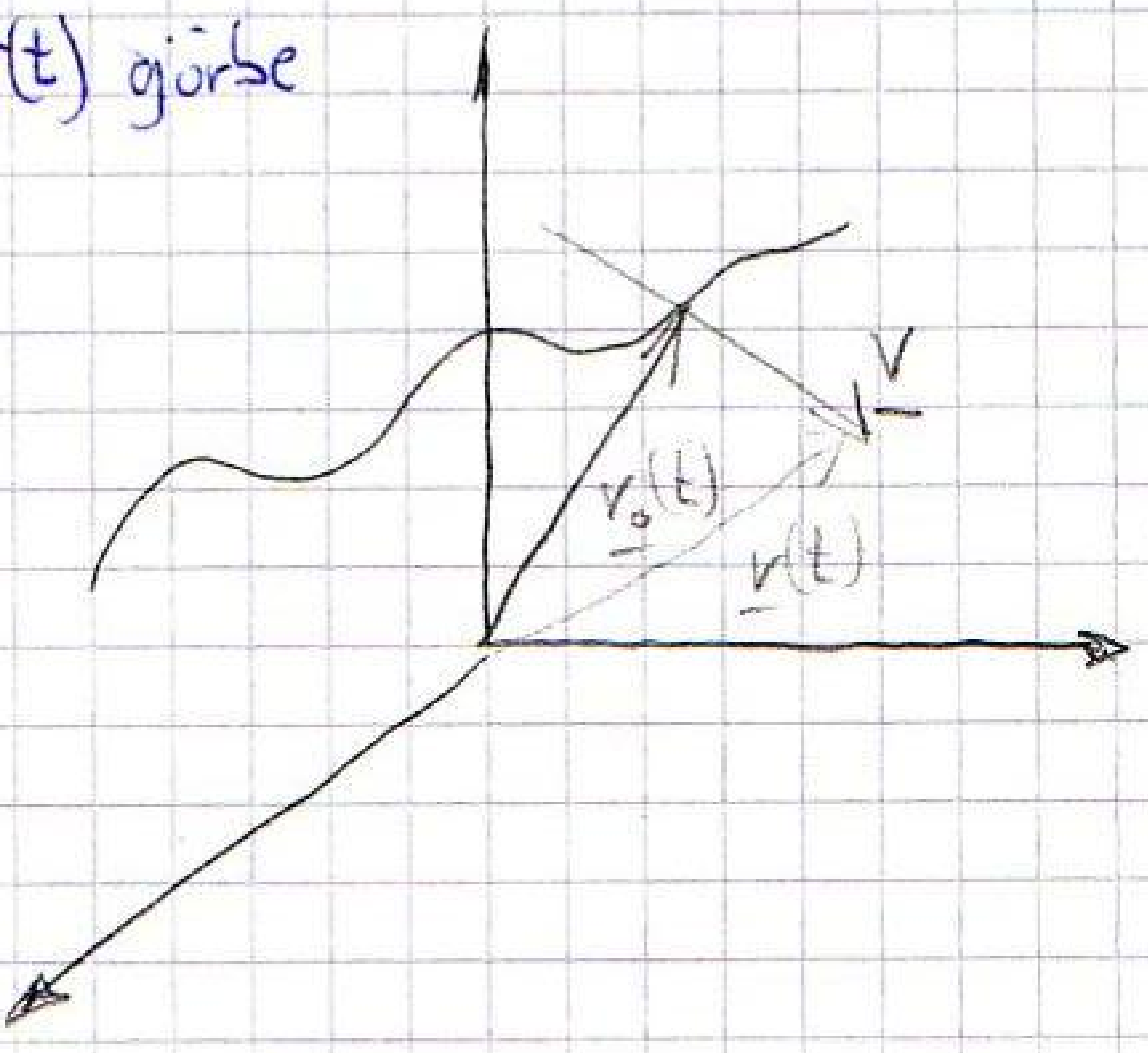
$$v\Rightarrow\varphi$$

$x = 1 + 5\sin u \cos v$

$y = 2 + 5\sin u \sin v$

$z = 3 + 5\cos u$

$\underline{r}(t)$ görbe



$\boxed{\text{érintő}}$

$\underline{r}(t) = \underline{r}_0(t) + t\underline{v}$

$\underline{v} = \dot{\underline{r}}(t)$

$\boxed{\text{pl}}$

$\underline{r}(t) = (t^2 - 2t,\ 3t - 5,\ -t^2 - 2)$

érintő $t_0 = 2$

$\underline{r}(t_0) = \underline{r}(2) = (0, 1, -6)$

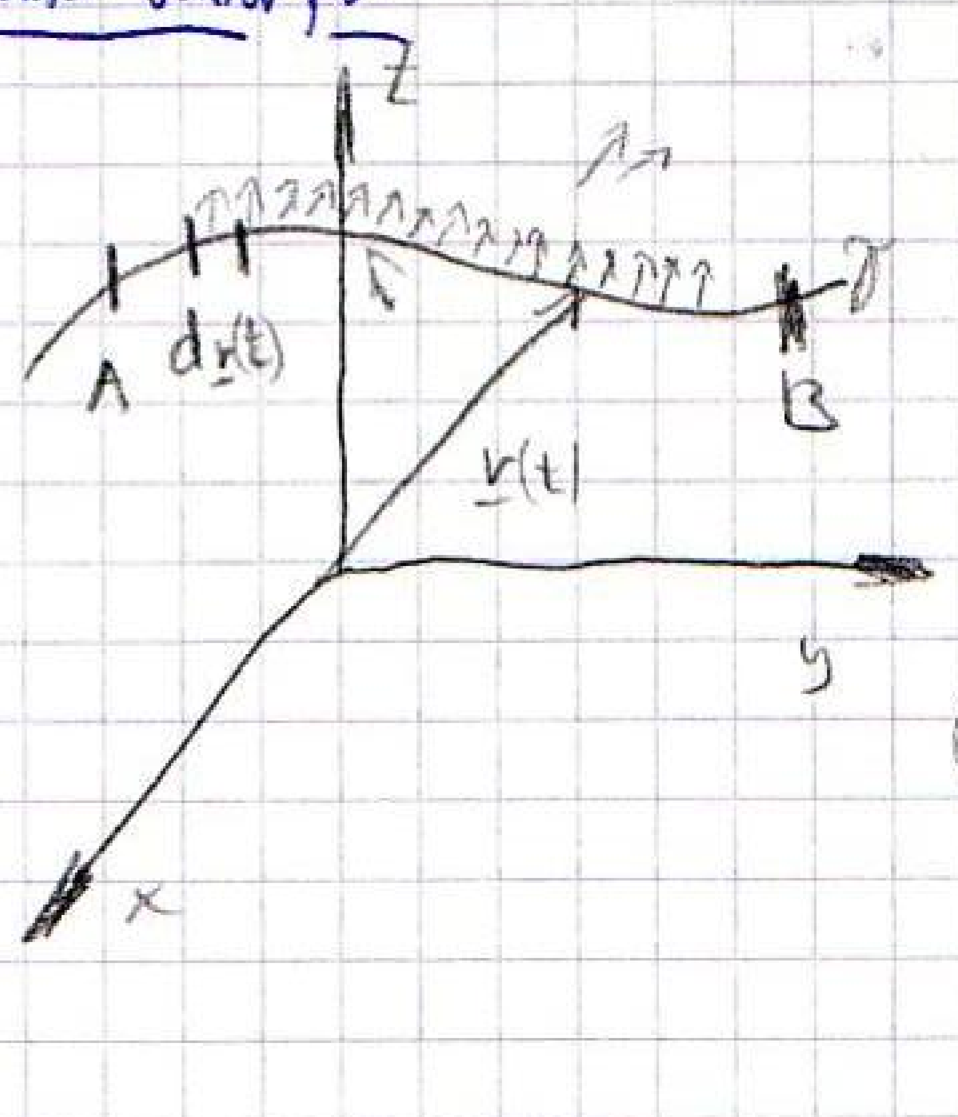
$\dot{\underline{r}}(t) = (2t - 2,\ 3,\ -2t)$

$\dot{\underline{r}}(2) = (2, 3, -4) = \underline{v}$

$$\underline{r}(t) = \underline{r}(2) + \dot{\underline{r}}(2)t = (0,1,-6) + (2t, 3t, -4t) = (\underbrace{2t}_{x}, \underbrace{3t+1}_{y}, \underbrace{-4t-6}_{z})$$

$\boxed{t = \frac{x}{2} = \frac{y-1}{3} = \frac{z+6}{-4}}$ egyenes egyenlete

vektor-vektor fv



$$\underset{(\gamma)}{\int_A^B} \underline{v}(\underline{r})\, d\underline{r}$$

$$\int_{A,\gamma}^{B} \underline{v}(\underline{r})\, d\underline{r} = \int_{t_1}^{t_2} \underline{v}(\underline{r}(t))\, \dot{\underline{r}}(t)\, dt$$

$\gamma : \underline{r} = \underline{r}(t)$

$$d\underline{r} = \frac{d\underline{r}}{dt}\, dt = \dot{\underline{r}}t\, dt$$