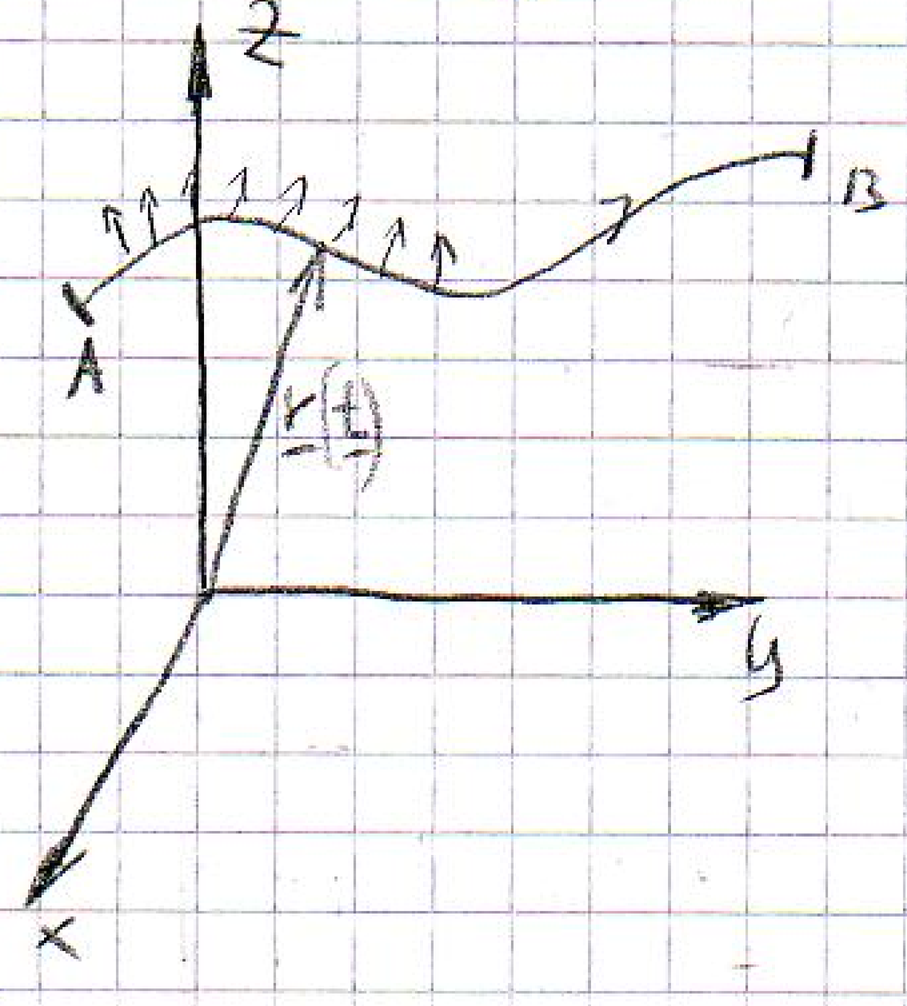
2007.03.29 csütörtök VI. Gyakorlat (7. hét)

Vektor-vektor fv-ek

$\underline{v}: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad \underline{v}(\underline{r})$



$\gamma: \underline{r} = \underline{r}(t) = (x(t), y(t), z(t))$

$t_1 \leq t \leq t_2 \qquad \underline{r}(t_1) = A$

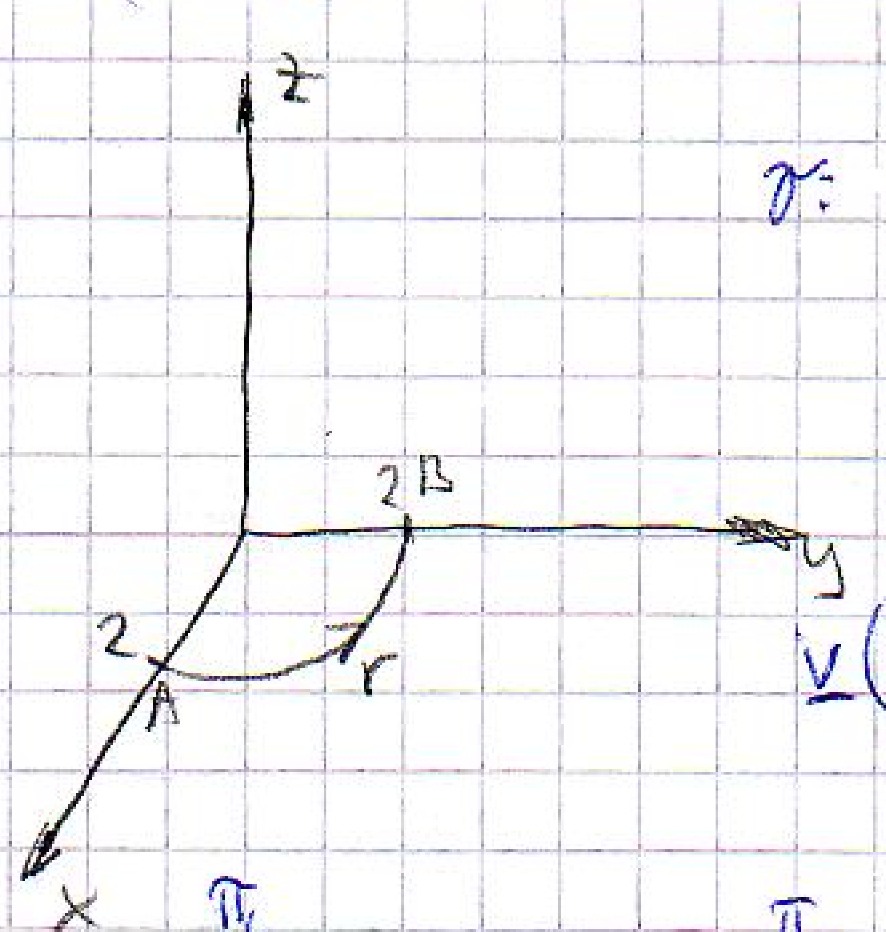
$\qquad\qquad\qquad\quad \underline{r}(t_2) = B$

$$\int_{A,\gamma}^{B} \underline{v}(\underline{r})\,d\underline{r} = \int_{t_1}^{t_2} \underline{v}(\underline{r}(t))\,\dot{\underline{r}}\,dt$$

$$d\underline{r} = \underbrace{\dot{\underline{r}}}_{\frac{d\underline{r}}{dt}}\,dt$$

$\underline{v}(\underline{r}) = yz\,\underline{i} + x^2\,\underline{j} + z\,\underline{k}$

$(1,0,2) \mapsto (0,1,2)$



$\gamma: \underline{r}(t) = 2\cos t\,\underline{i} + 2\sin t\,\underline{j}$

$0 \leq t \leq \frac{\pi}{2}$

$\dot{\underline{r}}(t) = -2\sin t\,\underline{i} - 2\cos t\,\underline{j}$

$\underline{v}(\underline{r}(t)) = 4\cos^2 t\,\underline{j}$

$$\int_{A,\gamma}^{B} \underline{v}(\underline{r})\,d\underline{r} = \int_0^{\pi/2} 8\underbrace{\cos^3 t}_{(1-\sin^2 t)\cos t}\,dt = 8\int_0^{\pi/2}\left(\cos t - \sin^2 t\cos t\right)dt = 8\left\{\left[\sin t\right]_0^{\frac{\pi}{2}} - \left[\frac{\sin^3 t}{3}\right]_0^{\frac{\pi}{2}}\right\} = \frac{16}{3}$$

$$\int_{n \neq 1} f^n f' = \frac{f^{n+1}}{n+1}$$

vektor-vektor fv

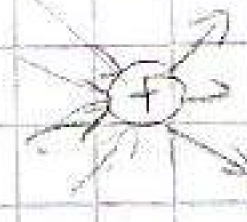
skalár-vektor fv

$f: \mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z) = f(\underline{r})$

$\text{grad } f(\underline{r}) = (f'_x, f'_y, f'_z)$ (skalár → vekt)

$\text{div } \underline{v}(\underline{r}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$ (forrásosság)



(skalárinv)

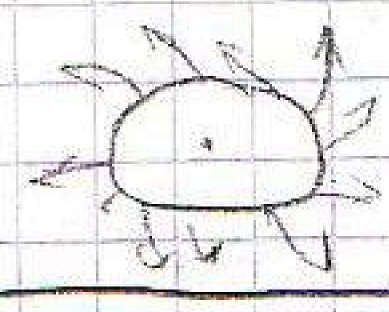
(vektor → skalár)

pl $\text{div } \underline{B} = 0$ (mágn monopólus nincs)

$$\text{rot } \underline{v}(\underline{r}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \underline{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \underline{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \underline{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

(vekt → vekt)

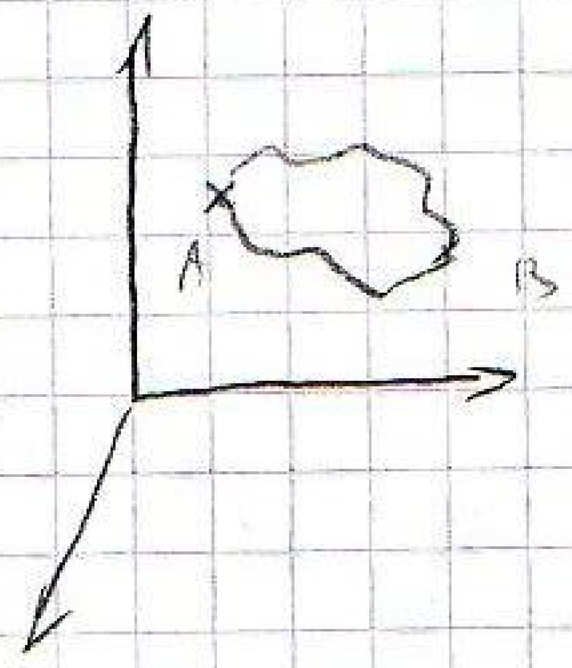
(örvényesség)



(vekt-inv)

$\underline{A}(\underline{r}): \mathbb{R}^3 \to \mathbb{R}^3$

$\underline{v} = \underline{v}(\underline{r})$



$\int_A^B \underline{v}(\underline{r})\, d\underline{r}$, ha független az úttól (csak A-tól és B-től függ)

$\Downarrow$

$\underline{v}(\underline{r})$ POTENCIÁLOS

$\Updownarrow$

$\oint \underline{v}(\underline{r})\, d\underline{r} = 0$

$\Updownarrow$

$\text{rot } \underline{v}(\underline{r}) = 0 \Leftrightarrow \exists\, u(\underline{r})$ potenciál fv

$\underline{v}(\underline{r}) = \text{grad } u$

$$\int_A^B \underline{v}(\underline{r})\, d\underline{r} = \int_A^B \overbrace{\underbrace{\text{grad } U\, d\underline{r}}_{\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)(dx, dy, dz)}}^{du} = [U]_A^B = U^B - U^A$$

∇ nabla op

$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\boxed{\begin{aligned} \text{grad } f &= \nabla f \\ \text{div } \underline{v} &= \nabla \cdot \underline{v} \\ \text{rot } \underline{v} &= \nabla \times \underline{v} \end{aligned}}$$

pl: $\text{rot grad } f = \nabla \times (\nabla f) = 0$

példa

$$\underline{v}(\underline{r}) = (2yz - x^2,\ 2xz + y^2,\ 2xy + z^2)$$

$$\left.\begin{array}{l} A = (2,1,3) \\ B = (-1,3,-2) \end{array}\right\} \text{ egyenes szakasz } \int_A^B \underline{v}(\underline{r})\, d\underline{r} = ?$$

① potenciálos-e a $\underline{v}$

$$\text{rot } \underline{v}(\underline{r}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz - x^2 & 2xz + y^2 & 2xy + z^2 \end{vmatrix} =$$

$$= \underline{i}(2x - 2x) - \underline{j}(2y - 2y) + \underline{k}(2z - 2z) = 0 \Rightarrow \text{potenciálos}$$

② potenciál fv.?

$$\underline{v}(\underline{r}) = \text{grad } U = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right)$$

$$\begin{cases} \frac{\partial U}{\partial x} = 2yz - x^2 \\ \frac{\partial U}{\partial y} = 2xz + y^2 \\ \frac{\partial U}{\partial z} = 2xy + z^2 \end{cases}$$

$$\Rightarrow U = \int (2yz - x^2)\, dx = 2xyz - \frac{x^3}{3} + C(y,z)$$

$$\frac{dU}{dy} = 2xz + \frac{\partial C(y,z)}{\partial y} = 2xz + y^2 \Rightarrow C(y,z) = \int y^2 dy = \frac{y^3}{3} + C(z)$$

$$U = 2xyz - \frac{x^3}{3} + \frac{y^3}{3} + C(z)$$

$$dU = 2xy + \frac{dC(z)}{dz} = 2xy + z^2 \Rightarrow C(z) = \int z^2 dz = \frac{z^3}{3} + C$$

$$U = 2xyz - \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} + C$$

nullpont

$$\int_A^B \underline{v}(\underline{r})\, d\underline{r} = U(B) - U(A) = \left(12 + \frac{1}{3} + 9 - \frac{8}{3}\right) - \left(12 - \frac{8}{3} + \frac{1}{3} + 9\right) = 0$$

$\Rightarrow$ A, B ekvipotenciális

példa| $\operatorname{div}\operatorname{rot}\underline{v} = ?\left(\frac{\partial^2 v_3}{\partial y\partial x} - \frac{\partial^2 v_2}{\partial z\partial y}\right) - \left(\frac{\partial^2 v_3}{\partial x\partial y} - \frac{\partial^2 v_1}{\partial z\partial y}\right) + \left(\frac{\partial^2 v_2}{\partial x\partial z} - \frac{\partial^2 v_1}{\partial y\partial z}\right) = 0$

(a)

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \underline{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \underline{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \underline{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

Young-Schwarz

v'_x és v'_y folyt $\Rightarrow v''_{xy} = v''_{yx}$

(b)

$$\nabla \cdot (\nabla \times \underline{v}) = 0$$

mert $\perp$ ∇-ra

példa|

$$\operatorname{div}(u \cdot \underline{v}) = \nabla(u\underline{v}) = (\nabla u)\cdot\underline{v} + u(\nabla\underline{v}) = \operatorname{grad} u\,\underline{v} + u \cdot \operatorname{div}\underline{v}$$

példa|

$$\operatorname{rot}(u\underline{v}) = \nabla \times (u\underline{v}) = (\nabla u) \times \underline{v} + u(\nabla \times \underline{v}) = \operatorname{grad} u \times \underline{v} + u \cdot \operatorname{rot}\underline{v}$$

példa|

$$\operatorname{div}\left(|\underline{r}| \operatorname{grad} \underbrace{\log|\underline{r}|^3}\right) = \operatorname{div}\left(|\underline{r}| \cdot \frac{3}{|\underline{r}|^2}\underline{r}\right) = \operatorname{div}\left(3\frac{\underline{r}}{|\underline{r}|}\right) = \frac{9}{|\underline{r}|} - \frac{3}{|\underline{r}|^3}\underbrace{(x^2+y^2+z^2)}_{|\underline{r}|^2} = \underline{\underline{\frac{6}{|\underline{r}|}}}$$

$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

$\frac{3}{2}\log(x^2+y^2+z^2) = f$

$|\underline{r}| = \sqrt{x^2+y^2+z^2}$

$|\underline{r}| = (x^2+y^2+z)^{3/2}$

$$\frac{df}{dx} = \frac{3}{2}\frac{1}{(x^2+y^2+z^2)}2x = \frac{3x}{|\underline{r}|^2}$$

$$\frac{df}{dy} = \frac{3y}{|\underline{r}|^2}$$

$$\frac{df}{dz} = \frac{3z}{|\underline{r}|^2}$$

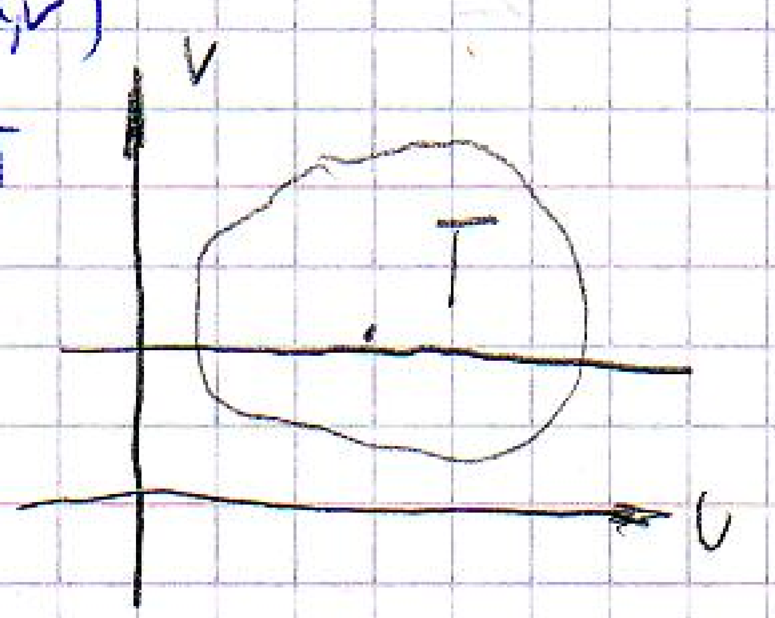
$$\frac{\partial}{\partial x}\left(\frac{3x}{|\underline{r}|}\right) = \frac{\partial}{\partial x}\left(\frac{3x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{3\sqrt{x^2+y^2+z^2} - 3x\frac{1}{2}(x^2+y^2+z^2)^{-1/2}2x}{x^2+y^2+z^2} =$$

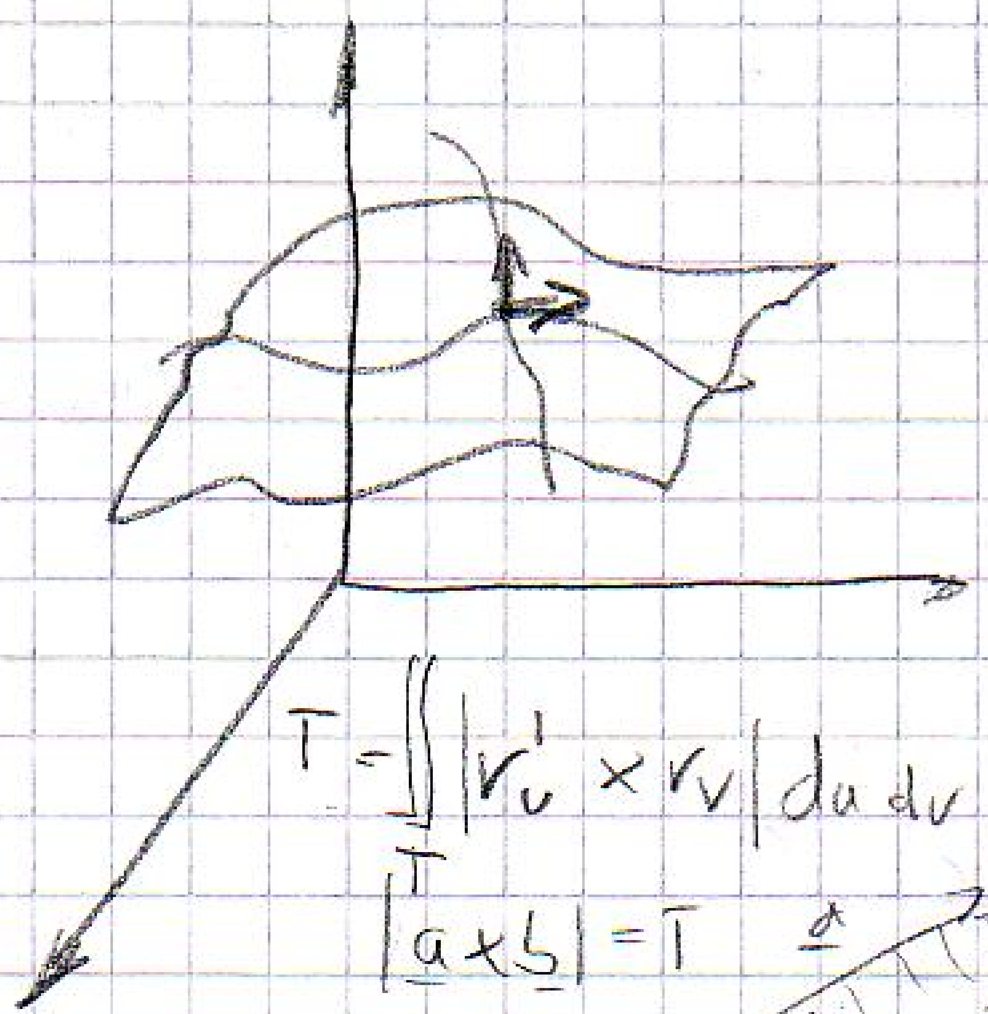
$$= \frac{3|\underline{r}| - \frac{3x^2}{|\underline{r}|}}{|\underline{r}|^2} = \frac{3}{|\underline{r}|} - \frac{3x^2}{|\underline{r}|^3}$$

Felszínszámítás

$\underline{r} = \underline{r}(u,v)$

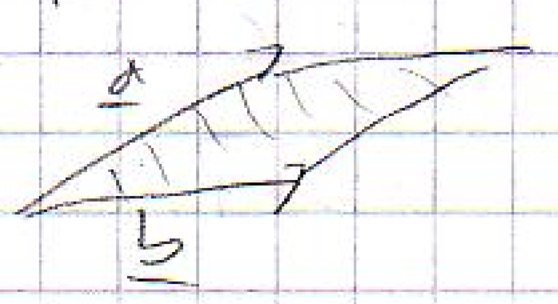
$(u,v) \in T$



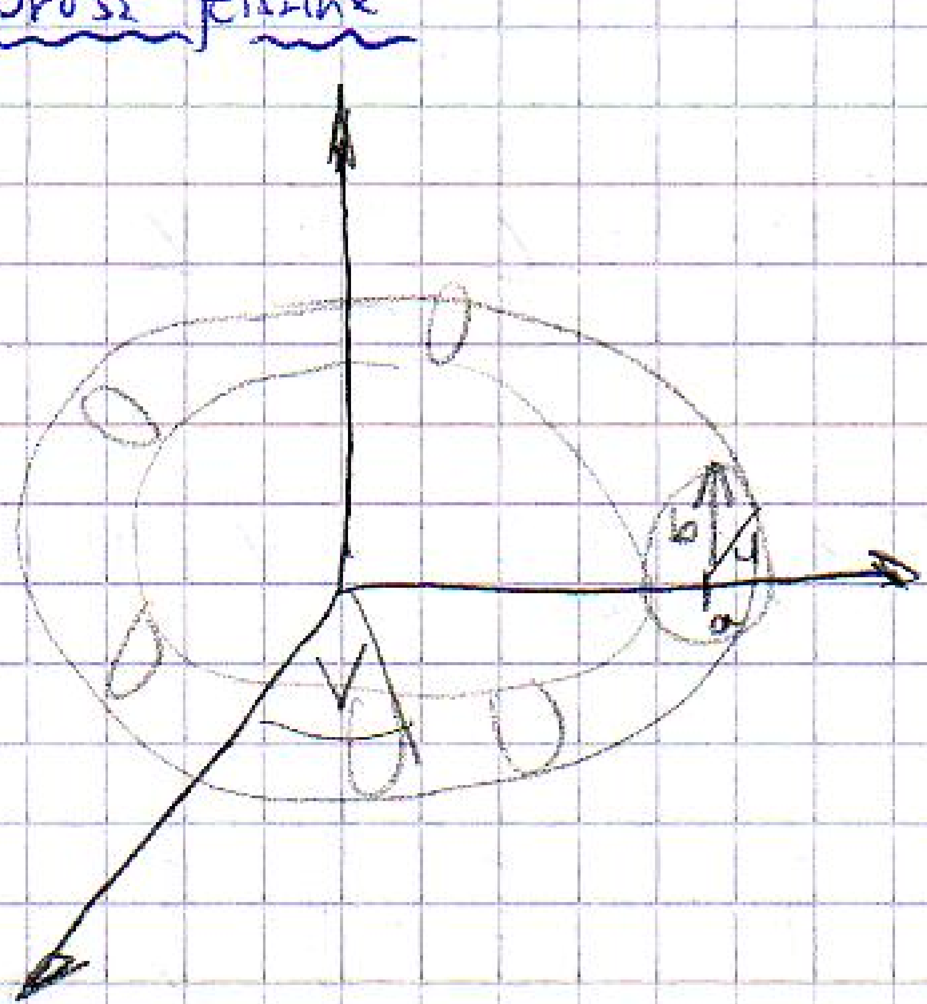


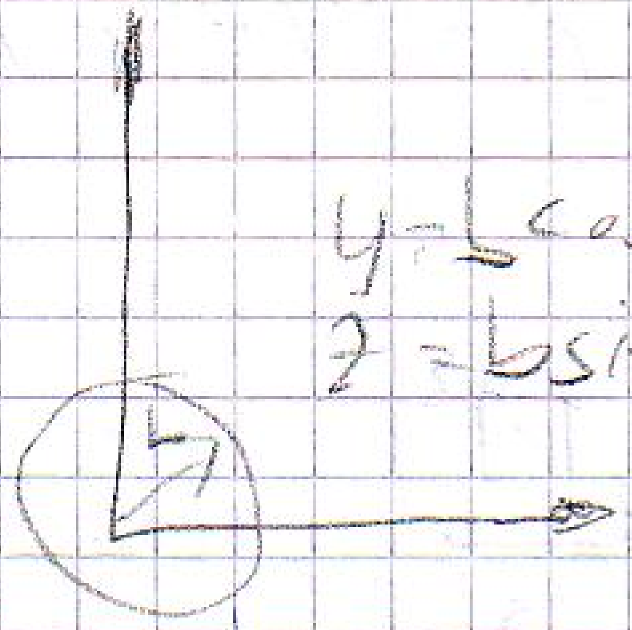
$$T = \iint_T |\underline{r}'_u \times \underline{r}'_v| \, du \, dv$$

$$|\underline{a} \times \underline{b}| = T$$



Tórusz felszíne





$y = b\cos u$

$z = b\sin u$

$$\underline{r}(u,v) = \underline{i}(a + b\cos u)\cos v + \underline{j}(a + b\cos u)\sin v + \underline{k}\, b \sin u$$

$$\underline{r}'_u = \underline{i}(-b \sin u \cos v) - \underline{j}(b \sin u \sin v) + \underline{k}\, b\cos u$$

$$\underline{r}'_v = -\underline{i}(a + b\cos u)\sin v + \underline{j}(a + b\cos u)\cos v$$

$$\underline{r}'_u \times \underline{r}'_v = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ -b\sin u\cos v & -b\sin u \sin v & b\cos u \\ -(a+b\cos u) & (a+b\cos u)\cos v & 0 \end{pmatrix} =$$

$$= \underline{i}\big(b\cos u\cos v(a+b\cos u)\big) + \underline{j}\, b\cos v(a+b\cos u)\sin v + \underline{k}\big(-b\sin u\cos v(a+b\cos u)\cos v -$$

$$-(a+b\cos u) b \sin u \sin^2 v$$

$$|\underline{r}'_u \times \underline{r}'_v| = b(a + b\cos u)$$

$$F = \int_0^{2\pi}\int_0^{2\pi} b(a+b\cos u)\,du\,dv = \int_0^{2\pi} \overbrace{\Big[bau + b\sin u\Big]_0^{2\pi}}^{2\pi ba} dv = 2\pi ba \big[v\big]_0^{2\pi} = \underline{\underline{4\pi^2 ab}}$$