13 Gyakorlat

2007.04.05.
CSÜTÖRTÖK

INTEGRÁL ÁTALAKÍTÓ TÉTELEK

(17) (50+)

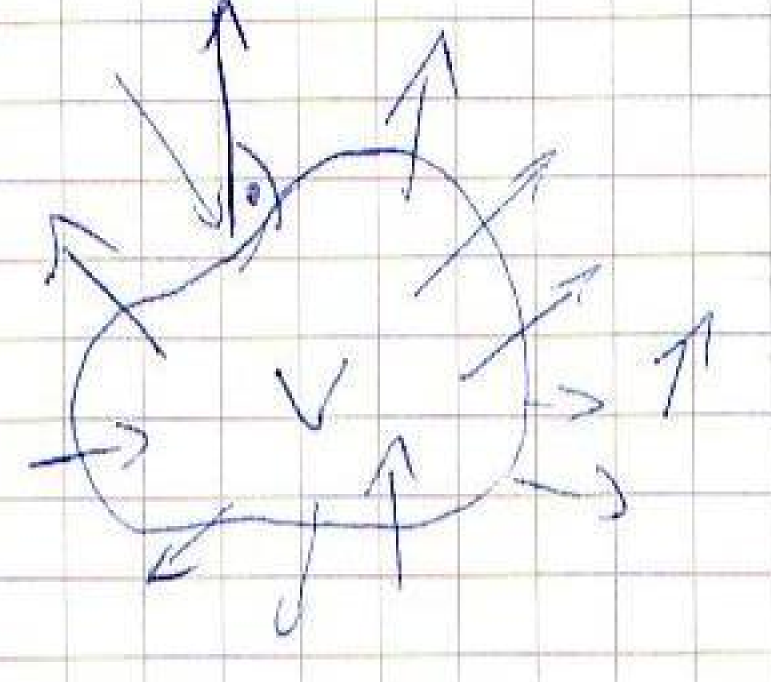
1) GAUSS - OSZTROGRADSZKIJ

adott pontban, a v felületre merőleges vektor

$$\iiint_V \operatorname{div} \underline{v}(\underline{r})\, dV = \oiint_F \underline{v}\, d\overbrace{\underline{F}}$$

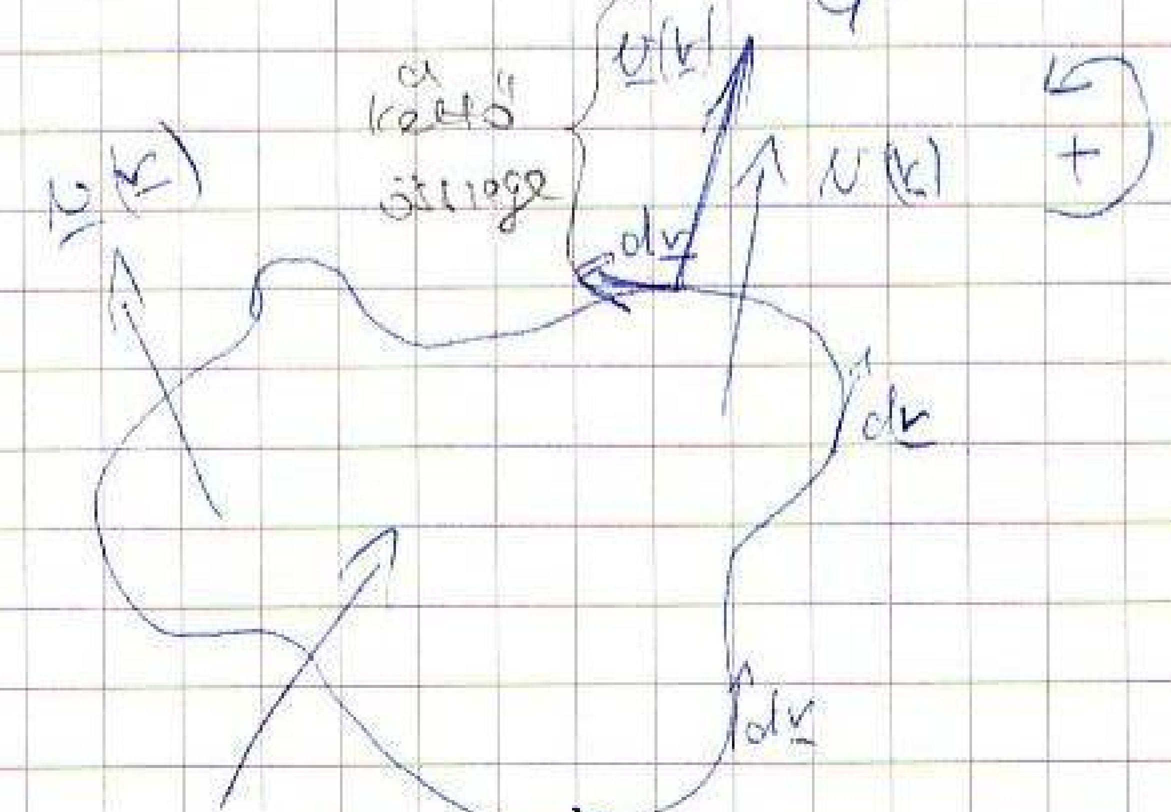
a felület kifelé mutató normálvektor

$\underline{n} = dF$



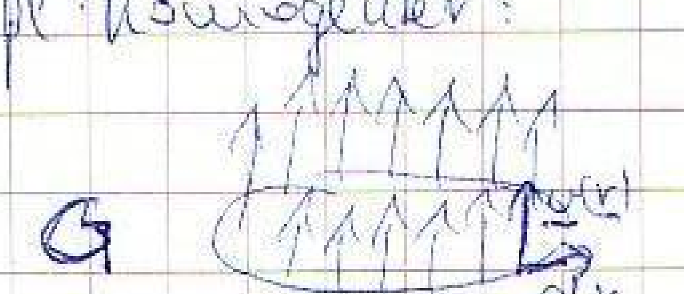
2) Stokes - tétel:

$$\iint_F \operatorname{rot} \underline{v}\, d\overbrace{\underline{F}}^{\text{normális}} = \oint_G \underline{v}\, d\underline{r}$$



$d\underline{r}$: minden pontban valahogy állnak

pl: homogén tér:



#1 GREEN FORMULA

$u_1, u_2 : \mathbb{R}^3 \to \mathbb{R}$

Laplace operator $\Delta = \nabla^2$

(1) $$\iiint_V \left(u_1 \Delta u_2 + \text{grad } u_1 \cdot \text{grad } u_2 \right) dV = \iint_F u_1 \cdot \text{grad} \cdot u_2 \, dF$$

$\underline{v}(\underline{r}) = u_1 \cdot \nabla u_2$

$\Delta = \nabla^2$

Laplace operator

$$\Delta u(\_) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

/div/

$$\nabla^2 u = \nabla \cdot (\nabla u)$$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \qquad \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)$$

$$\text{div } \underline{v}(\underline{r}) = \nabla \cdot \underline{v}(\underline{r}) = \nabla(u_1 \nabla u_2) = \nabla u_1 \cdot \nabla u_2 + u_1 \cdot \nabla(\nabla u_2)$$

$$\text{grad} u_1 \cdot \text{grad} u_2 + u_1 \Delta u_2$$

#2 GREEN FORMULA /az előzőből (#1) következik/

$$\iiint_V \left(U_1 \Delta U_2 - U_2 \Delta U_1 \right) dV = \iint_F \left(U_1 \operatorname{grad} U_2 - U_2 \operatorname{grad} U_1 \right) d\underline{F} \quad *$$

$U_1 \leftrightarrow U_2$ kicserélésével

$$\iiint_V \left(U_2 \Delta U_1 + \operatorname{grad} U_2 \operatorname{grad} U_1 \right) dV = \iint_F U_2 \operatorname{grad} U_1 \, d\underline{F} \quad **$$

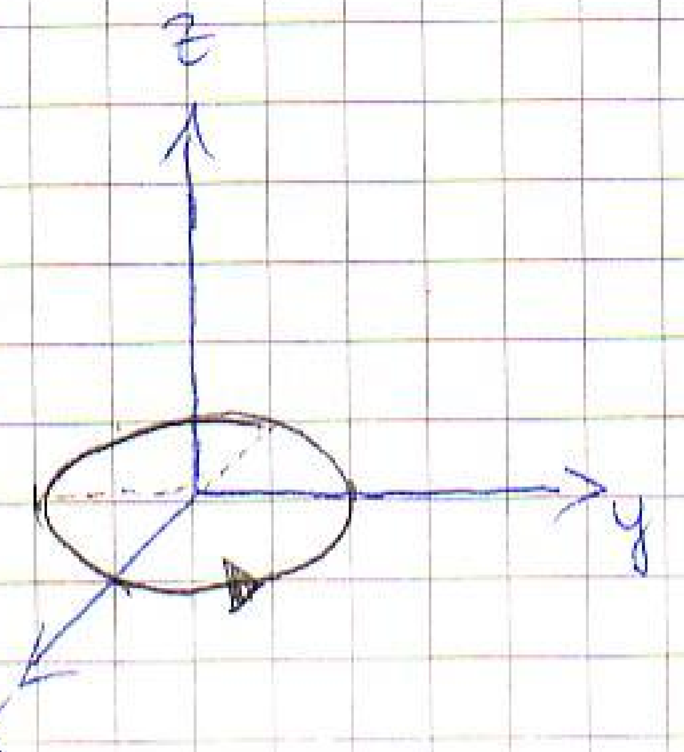
$(* - **)$ és már kész is.

$\boxed{\text{PÉLDA 2}}$

Biz be, hogy:

$$\oint_{\substack{x^2+y^2=2 \\ z=0}} \underbrace{\frac{\ln|\underline{r}|}{|\underline{r}|^2}\,\underline{r}}_{\underline{v}(\underline{r})} \cdot d\underline{r} = \emptyset$$

Stokes-tétel alkalmazás



$\underline{r} = (x, y, z)$

$|\underline{r}| = \sqrt{x^2+y^2+z^2}$

$$\underline{v}(\underline{r}) = \frac{\ln\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}(x, y, z) = \frac{\ln(x^2+y^2+z^2)}{2\cdot(x^2+y^2+z^2)}(x, y, z)$$

$$\text{rot } \underline{v}(\underline{r}) = \nabla \times \underline{v}(\underline{r}) = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x \cdot \ln(x^2+y^2+z^2)}{2(x^2+y^2+z^2)} & \frac{y \ln(x^2+y^2+z^2)}{2\cdot(x^2+y^2+z^2)} & \frac{z \ln(x^2+y^2+z^2)}{2(x^2+y^2+z^2)} \end{pmatrix} =$$

ezt kell kiszámolni!

Viszont nem vagyunk ügyesek...

Később kiszámoljuk ügyesen is

folyt.

folyt

$$= \underline{i}\left(\frac{\frac{z}{x^2+y^2+z^2}\cdot 2y\cdot 2(x^2+y^2+z^2) - 4y\cdot z\cdot \ln(x^2+y^2+z^2)}{4(x^2+y^2+z^2)^2} - \right.$$

$$\left. - \frac{y\cdot\frac{1}{x^2+y^2+z^2}\cdot 2z\cdot 2(x^2+y^2+z^2) - 4zy\,(x^2+y^2+z^2)}{4(x^2+y^2+z^2)^2}\right)$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\emptyset}$$

$$= \underline{i}(0) - \underline{j}(0) + \underline{k}(0) = \emptyset \rightarrow \text{rot } \underline{v}(\underline{r}) = \emptyset$$

Tehát: $\oint \frac{\ln|\underline{r}|}{|\underline{r}|^2}\,\underline{r}\,d\underline{r} = 0 \Leftarrow \text{rot } \underline{v}(\underline{r}) = \emptyset$

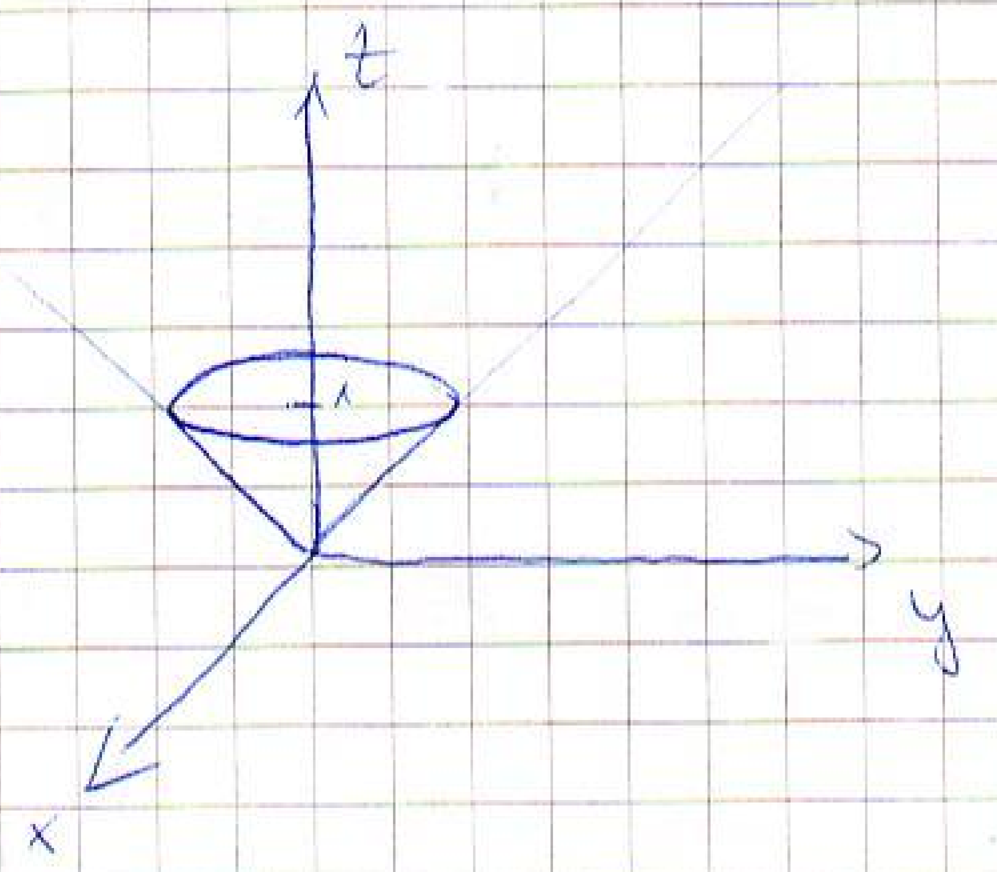
Példa 3

$$\underline{v}(\underline{r}) = (x^3, y^3, z^3)$$

kúpfelület: $\left.\begin{aligned} 9z^2 &= x^2 + y^2 \\ z &= 1 \end{aligned}\right\} \underline{F}$

$$\oiint_F \underline{v}(\underline{r})\, d\underline{F} = ? \quad \Big/ \Longrightarrow \iiint_V \operatorname{div} \underline{v}(\underline{r})\, dV \Big/$$

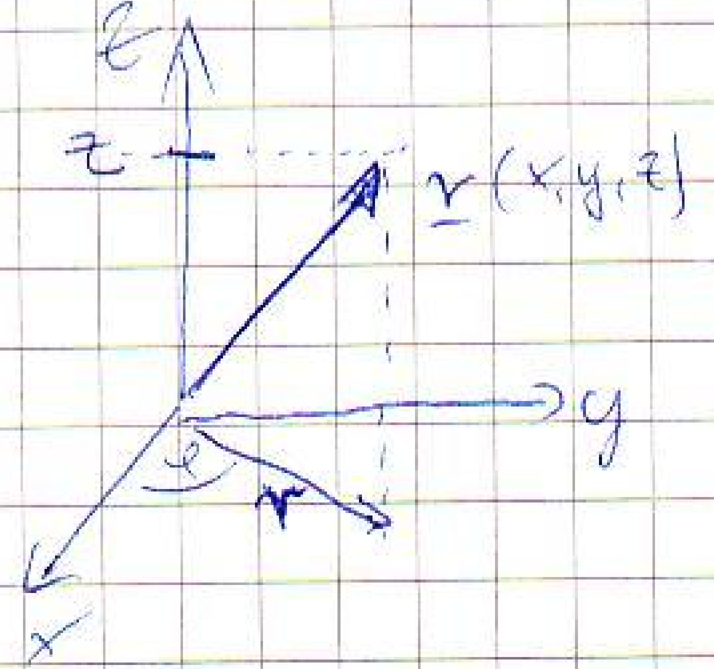
tényleg zárt-e a felület? /a tétel csak zárt esetekre érvényes/



$$\operatorname{div} \underline{v}(\underline{r}) = \nabla \cdot \underline{v}(\underline{r}) = 3x^2 + 3y^2 + 3z^2 = 3\cdot(x^2+y^2+z^2) =$$

$$= 3 \cdot |\underline{r}|^2$$

HENGERKOORDINÁTÁK bevezetése!

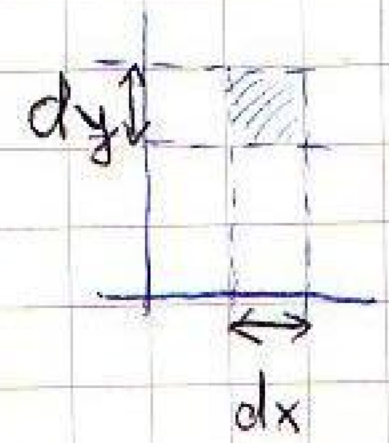


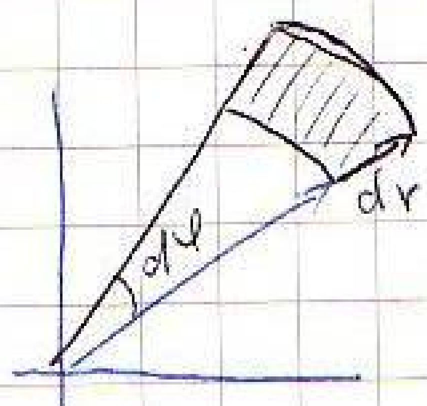
$x = r \cdot \cos\varphi$

$y = r \cdot \sin\varphi$

$z = z$

$$\iint f(x,y)\,dx\,dy \Longleftrightarrow \iint f(r,\varphi)\,dr\,d\varphi$$





jakobi-determináns

$$dx\,dy = r\,dr\,d\varphi$$

$$dx\,dy\,dz \Longleftrightarrow r\cdot dr\,d\varphi\,dz$$

koordináták bevezetése:

$x = r\cdot\cos\varphi$

$y = r\cdot\sin\varphi$

$z = z$

$$\boxed{0 \le \varphi \le 2\pi}$$

feltétel: $9z^2 = r^2\cdot\cos^2\varphi + r^2\cdot\sin^2\varphi = r^2$

$$\boxed{0 \le r \le 3z}$$

$$\boxed{0 \le z \le 1}$$

$$\operatorname{div} \underline{v}(\underline{r}) = 3r^2 + 3z^2$$

jakobi

$$\underset{(z)}{\int_0^1}\underset{(\varphi)}{\int_0^{2\pi}}\underset{(r)}{\int_0^{3z}} (3r^2+3z^2)\, r\, dr\,d\varphi\,dz = \int_0^1\int_0^{2\pi} \underbrace{\left[\frac{3r^4}{4} + \frac{3z^2r^2}{2}\right]_0^{3z}}_{\left(\frac{243z^4}{4}+\frac{27z^4}{2}\right)-0 \;=\; \frac{297\cdot z^4}{4}} d\varphi\,dz =$$

$$= \int_0^1 \underbrace{\left[\frac{297\cdot z^4}{4}\varphi\right]_0^{2\pi}}_{\frac{297\cdot z^4}{2}\pi} dz = \left[\frac{297\cdot z^5}{10}\pi\right]_0^1 = \underline{\underline{\frac{297\pi}{10}}}$$

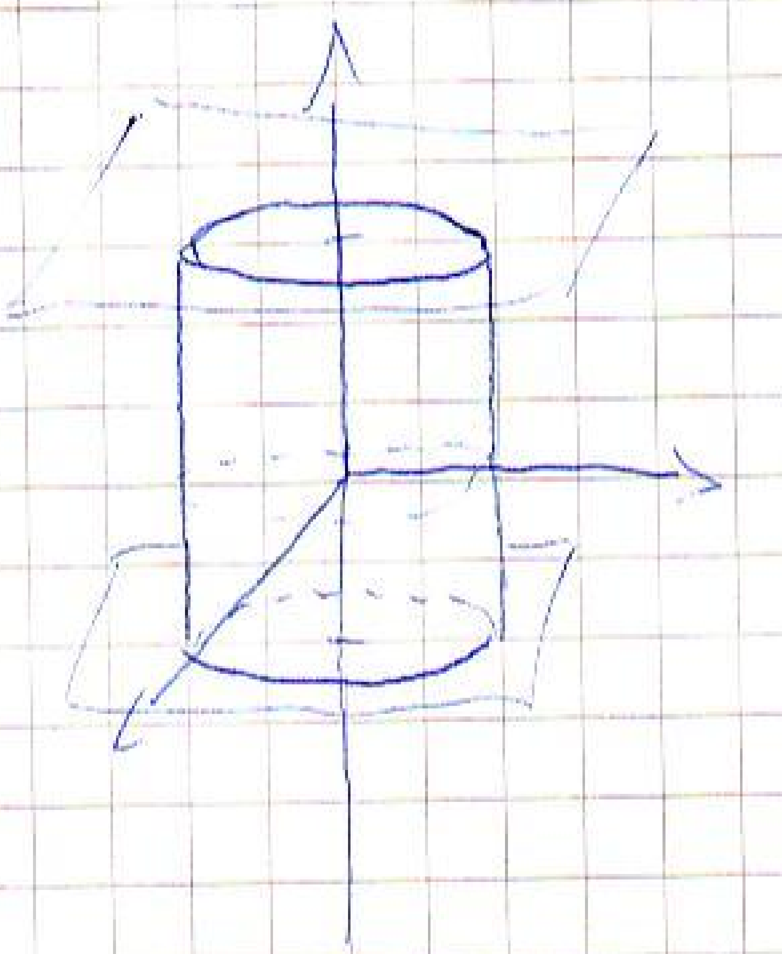
$\boxed{\text{Példa (4)}}$

$\underline{v}(\underline{r}) = (x, 2y, 3z)$

Körhenger: $\left.\begin{array}{l} x^2 + y^2 = 9 \\ -2 \le z \le 3 \end{array}\right\} F$ $\quad \oint\!\!\oint_F \underline{v}(\underline{r})\, d\underline{F} = ?$

$$\oint\!\!\oint \underline{v}(\underline{r})\, d\underline{F} = ?$$

$$\oint\!\!\oint = -\iiint_V$$



$$\iiint_V \operatorname{div} \underline{v}(\underline{r})\, dV = 6 \underbrace{\iiint_V dV}_{\text{Henger térfogata}} = 6 \cdot 45\pi = \underline{\underline{270\pi}}$$

$\underline{\underline{270\pi}}$

$$\operatorname{div} \underline{v}(\underline{r}) = \nabla \underline{v}(\underline{r}) = 1 + 2 + 3 = \underline{6}$$

$$\iiint_V \rho(x,y,z)\, dV \rightarrow \text{tömege a testnek}$$

ha $\rho(x,y,z) = 1$

$$\Rightarrow V = \iiint dV$$

$$V = r^2\pi \cdot m = 3^2\pi \cdot 5 = \underline{45\pi}$$

magyarázat:

∇ vektor $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

a) $\nabla \cdot u(\underline{r}) = \left(\frac{\partial u(r)}{\partial x}, \frac{\partial u(r)}{\partial y}, \frac{\partial u(r)}{\partial z}\right)$

$\nabla : u(\underline{r}) \longmapsto \text{grad } u(\underline{r})$

b) $\nabla \cdot \underline{v}(\underline{r}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \text{div } \underline{v}(\underline{r})$

$\underline{v}(\underline{r}) = \left(v_1(\underline{r}), v_2(\underline{r}), v_3(\underline{r})\right)$

$\nabla : \underline{v}(\underline{r}) \longmapsto \text{div } \underline{v}(\underline{r})$