2007.04.12.
CSÜTÖRTÖK

13 Gyakorlat

(1)

$$y' = f\left(\frac{y}{x}\right)$$

$$xy^2y' = x^3 + y^3 \qquad x \neq 0$$

$$\left(\frac{y}{x}\right)^2 y' = 1 + \left(\frac{y}{x}\right)^3 \qquad y \neq 0$$

$$y' = \frac{1 + \left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2}$$

új változó: $u := \frac{y(x)}{x}$

$$y = u \cdot x$$

$$y' = u'x + u$$

$$u'x + u = \frac{1+u^3}{u^2}$$

$$u'x = \frac{1+u^3}{u^2} - u = \frac{1}{u^2}$$

$$\frac{du}{dx}x = \frac{1}{u^2}$$

$$u^2\,du = \frac{dx}{x}$$

$$\int u^2\,du = \int \frac{dx}{x}$$

$$\frac{u^3}{3} = \ln|x| + \ln C$$

$$\left(\frac{y}{x}\right)^3 = 3 \cdot \ln Cx \quad \rightarrow \text{implicit alak!}$$

DF: $y=0$ megoldás?

$\left.\begin{array}{l} x=0 \\ y=0 \end{array}\right\}$ is megoldás!!!

(2) $y' = f(ax+by+c)$

$$3y' = 3(4x+3y-2)^2 - (4x+3y+4) = 3(4x+3y-2)^2 - (4x+3y-2) - 6$$

$$u = (4x+3y-2)$$

$$3y = u - 4x + 2$$

$$3y' = u' - 4$$

$$u' - 4 = 3u^2 - u - 6$$

$$u' = 3u^2 - u - 2$$

$$\frac{du}{dx} = 3u^2 - u - 2$$

$$\int \frac{du}{3u^2 - u - 2} = \int dx$$

$$\int \frac{du}{\underbrace{\left(\sqrt{3}u - \frac{1}{2\sqrt{3}}\right)^2}_{3u^2 - u + \frac{1}{12}} - \frac{25}{12}}$$

új változó:

$$x = \operatorname{ch} y$$

$$\frac{dx}{dy} = \operatorname{sh} y \qquad dx = \operatorname{sh} y \, dy$$

$$\int \frac{dx}{x^2-1} = \int \frac{1}{\operatorname{ch}^2 y - 1} \operatorname{sh} y \, dy = \int \frac{1}{\operatorname{sh} y} dy$$

... folytatás ...

③ LW, diff. egyenlet:

$$y' \sin x + \cos x = 1 + y$$

(általános alak: $y' + a(x)\,y = h(x)$)

$$\sin x \neq 0$$

$$y' - \frac{y}{\sin x} = \frac{1-\cos x}{\sin x}$$

H: $y' - \frac{y}{\sin x} = 0$

$$\frac{dy}{dx} = \frac{y}{\sin x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{\sin x}$$

/ $\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$ /

$\ln|y| =$

⋮

(?)

$u = \sin x$

$u' = \cos x$

$du = \underbrace{\cos x}_{\sqrt{1-u^2}}\, dx \implies dx = \frac{du}{\sqrt{1-u^2}}$

Feladat: 4

$$y'' - 4y' + 3y = 0$$

$y(0) = 6$
$y'(0) = 10$

$$y = e^{\lambda x}$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\underbrace{\lambda_1 = 3 \qquad \lambda_2 = 1}_{1x\text{-es gyökök}}$$

$$y_H = C_1 \cdot e^{3x} + C_2 \cdot e^{x}$$

$$y_H' = 3 \cdot C_1 \cdot e^{3x} + C_2 \cdot e^{x}$$

$$6 = C_1 + C_2$$

$$10 = 3C_1 + C_2$$

$$C_1 = 2$$

$$\underline{\underline{C_2 = 4}}$$

5 Laplace

$\dot{x} = x - 3y$ t.e. $\begin{cases} x(0) = 1 \\ y(0) = -1 \end{cases}$

$\dot{y} = y + 3x$

$x \xleftrightarrow{\mathcal{L}^{-1}} X(z)$

$\dot{x} \longrightarrow zX - x(0) = zX - 1$

$y \longrightarrow Y(z)$

$\dot{y} \longrightarrow zY - y(0) = zY + 1$

$zX - 1 = X - 3Y \longrightarrow Y = \dfrac{X(1-z)+1}{3}$

$zY + 1 = Y + 3X$

$$z\frac{X(1-z)+z}{3} + 1 = \frac{X(1-z)+1}{3} + 3X$$

$$X\Big(\underbrace{\frac{z(1-z)}{3} - \frac{1-z}{3} - 3}_{\frac{(1-z)(z-1)-9}{3}}\Big) = \underbrace{-1 + \frac{1}{3} - \frac{z}{3}}_{\frac{-2-z}{3}}$$

$$X = \frac{-2-z}{\underbrace{(1-z)(z-1)}_{-(z-1)^2} - 9} = \frac{z+2}{(z-1)^2+9} =$$

$$= \underbrace{\frac{z-1}{(z-1)^2+9}}_{\cos t} + \underbrace{\frac{3}{(z-1)^2+9}}_{e^t \cdot \sin 3t}$$

⑥ Integráló tényező keresése a köv. diff. egyenlethez

$u(x,y) = \varphi(x)$ integráló tényező

(1,1) ponton átmenő

$$(1 - x^2 y) + y' x^2 (y - x) = 0$$

$$(1 - x^2 y)\,dx + x^2 (y - x)\,dy = 0 \quad / \varphi(x)$$

$$\underbrace{\varphi(x)\,(1 - x^2 y)}_{h(x,y)}dx + \underbrace{\varphi(x)\,\overbrace{x^2 (y - x)}^{x^2y - x^3}}_{g(x,y)}dy = 0$$

egzakt: $\dfrac{\partial h}{\partial y} = \dfrac{\partial g}{\partial x}$

$$\frac{\partial h}{\partial y} = -x^2\,\varphi(x)$$

|| egyenlő

$$\frac{\partial g}{\partial x} = \varphi'(x^2 y - x^3) + \varphi(x)\,(2xy - 3x^2)$$

diff. egyenletet kapunk

$$\varphi(x)\Big(\underbrace{-x^2 - 2xy + 3x^2}_{\underbrace{2x^2 - 2xy}_{2x(x-y)}}\Big) = \underbrace{\varphi'(x)}_{\frac{d\varphi}{dx}}\,\underbrace{(x^2 y - x^3)}_{x^2(y-x)}$$

$$2x\,\varphi(x) = -x^2\,\frac{d\varphi}{dx}$$

$$\frac{d\varphi}{\varphi} = -\frac{2\,dx}{x}$$

$$\int \frac{d\varphi}{\varphi} = \int -\frac{2\,dx}{x} \quad \text{↳ folyt.}$$

$$\ln|\varphi| = -2\ln|x|$$

$$\varphi(x) = \frac{1}{x^2}$$

Tehát az integráló tényező: $\frac{1}{x^2}$

$$\frac{1-x^2y}{x^2}\,dx + (y-x)\,dy = 0$$ Ez már biztos, hogy egzakt

$\Downarrow$

$\exists\, F(x,y) = 0$ megoldás:

$$\frac{\partial F}{\partial x} = \frac{1-x^2y}{x^2} = x^{-2} - y$$

$$\frac{\partial F}{\partial y} = y - x$$

$$F(x,y) = \int (x^{-2} - y)\,dx = -\frac{1}{x} - yx + c(y)$$

$$\frac{\partial F}{\partial y} = 0 - x + \frac{dc}{dy} = y - x$$

$$\frac{dc}{dy} = y$$

$$\int dc = \int y\,dy$$

$$c = \frac{y^2}{2} + C$$

Tehát a megoldás: $F(x,y) = -\frac{1}{x} - yx + \frac{y^2}{2} + K = 0$

$x = 1$

$y = 1$

$-1 - 1 + \frac{1}{2} + K = 0$

$K = \cancel{\frac{1}{4}}\ 1{,}5$

Feladat 7

$y' = -y^2 \cos x \qquad y(\pi) = -1$

szétvál. vált. diff. e.

(?) $\dfrac{dy}{y^2} = -\cos x \, dx$

(?) $\displaystyle\int \frac{dy}{y^2} = -\int \cos x \, dx$

$-\dfrac{1}{y} = -\sin x + C \qquad y \neq 0$

$y = \dfrac{1}{\sin x + C}$

$-1 = \dfrac{1}{\sin \pi + C} \Rightarrow \underline{\underline{C = -1}}$

(9) <u>vonalmenti integrál</u>

$$\int \underline{v}(\underline{r})\, d\underline{r} = \int_{\Gamma} (y^2 dx + x^2 dy) \quad \text{vonalintegrál}$$

$$\Gamma : \Big(\underbrace{a\cdot\cos(t)}_{x(t)}, \underbrace{b\cdot\sin(t)}_{y(t)}, \emptyset\Big) \qquad t\in[0,\pi]$$

$$\underline{v}(\underline{r}) = (y^2, x^2, 0) \longrightarrow \underline{v}(\underline{r}(t)) = (b^2\cdot\sin^2 t,\ a^2\cdot\cos^2 t,\ 0)$$

$$d\underline{r} = (dx, dy, dt) = \underline{\dot r}(t)\, dt = (-a\sin(t),\ b\cos(t),\ 0)$$

$$\int_{\Gamma} \underline{v}(\underline{r}(t))\, \underline{\dot r}(t)\, dt = \int_0^{\pi} \big(b^2\sin^2(t)(-a\sin t) + a^2\cos^2(t)\ b\cos(t)\big)\, dt =$$

$$= \int_0^{\pi} \left(-ab^2\cdot\sin^3 t + a^2 b\cdot\cos^3 t\right) dt =$$

<u>Segég</u>:

$$\int \sin^3 t\, dt = \int \sin(1-\cos^2 t)\, dt =$$

$$= \int \sin(t)\, dt - \int \sin t \cos^2 t\, dt = -\cos t +$$

$$+ \frac{\cos^3 t}{3} + C$$

$$\int f' f^{\alpha} = \frac{f^{\alpha+1}}{\alpha+1} \quad \alpha \neq -1 \quad \text{volt}$$

Nieg:

$$\int \cos^3 t\,dt = \int \cos t\,(1-\sin^2 t)\,dt =$$

$$= \int \cos t\,dt - \int \cos t \sin^2 t\,dt$$

$$\sin(t) - \frac{\sin^3 t}{3}$$

$$= \int_0^{\pi} \left(-ab^2 \sin^3 t + a^2 b \cos^3 t\right) dt =$$

$$= -ab \underbrace{\left[-\cos t + \frac{\cos^3 t}{3}\right]_0^{\pi}}_{1-\frac{1}{3}+1-\frac{1}{3} = 2-\frac{2}{3}} + a^2 b \underbrace{\left[\sin(t) - \frac{\sin^3 t}{3}\right]_0^{\pi}}_{\varnothing} =$$

$$= -\frac{4}{3} ab^2$$

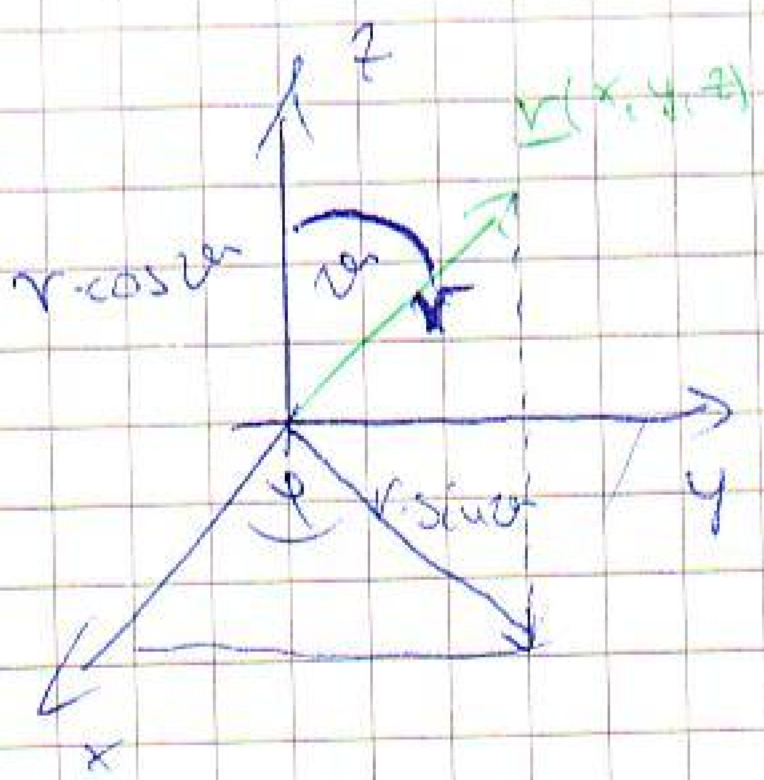
(13) $\underline{v}(\underline{r}) = x^3 \cdot \underline{i} + y^3 \underline{j} + z^3 \underline{k}$

$$\oiint_A \underline{v}(\underline{r})\, d\underline{F}$$

$A: \; x^2 + y^2 + z^2 = 4$

$$\iiint_V \operatorname{div} \underline{v}(\underline{r})\, dV$$

$\operatorname{div} \underline{v}(\underline{r}) = \nabla \cdot \underline{v}(\underline{r}) = 3x^2 + 3y^2 + 3z^2$



$(x, y, z) \longrightarrow (r, \vartheta, \varphi)$

$x = r \cdot \sin\vartheta \cdot \cos\varphi$

$y = r \cdot \sin\vartheta \cdot \sin\varphi$

$z = r \cdot \cos\vartheta$

Jacobi:

$|J| = r^2 \cdot \sin\vartheta$

$$J = \left| \frac{\partial(x, y, z)}{\partial(r, \vartheta, \varphi)} \right|$$

$\operatorname{div} \underline{u}(\underline{r}) =$

$= 3x^2 + 3y^2 + 3z^2 =$

$= 3\left(r^2 \sin^2\vartheta \cos^2\varphi + r^2 \sin^2\vartheta \cdot \sin^2\varphi + r^2 \cos^2\vartheta \right)$

$= 3r^2$

Tehát:

$$\int_{\substack{0\\ \varphi}}^{2\pi}\int_{\substack{0\\ \vartheta}}^{\pi}\int_{\substack{0\\ r}}^{2} \operatorname{div} \underline{u}(\underline{r})\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} 3r^2 \cdot r^2 \cdot \sin\vartheta \, dr\, d\vartheta\, d\varphi =$$

$$= \underbrace{\left[3\frac{r^5}{5}\right]_0^2}_{\frac{3\cdot 2^5}{5}} \cdot \underbrace{\left[-\cos\vartheta\right]_0^{\pi}}_{2} \cdot \underbrace{\left[\varphi\right]_0^{2\pi}}_{2\pi} = \frac{3\cdot 2^7}{5}\cdot \pi$$