2007.04.26 csütörtök

IX Gyakorlat (11. hét)

Komplex fv.-ton

$f: \mathbb{C} \mapsto \mathbb{C}$

$\boxed{I}$ $z = x+iy \mapsto f(z) = u(x,y) + i \cdot v(x,y)$

$Re = ?$
$Im = ?$

$\left[z \cdot \bar{z} = |z| \in \mathbb{R} \right]$

① $f(z) = \frac{1}{\bar{z}} = \frac{1}{x-iy} = \frac{1}{x-iy} \cdot \frac{x+iy}{x+iy} = \frac{x+iy}{x^2+y^2} = \underbrace{\frac{x}{x^2+y^2}}_{u(x,y)} + i \underbrace{\frac{y}{x^2+y^2}}_{v(x,y)}$

② $f(z) = z \cdot \bar{z} = \underbrace{x^2+y^2}_{U(x,y)} + i \cdot \underbrace{\phi}_{v(x,y)}$

③ $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x (\cos y + i \sin y) = e^x \cos y + i e^x \sin y$

$\left[\text{euler form.: } e^{iz} = \cos z + i \sin z \right]$ $\left[\text{Biz: Taylor-sor} \right]$

④ $f(z) = \sin z$ $\left[\begin{array}{l} e^{-iz} = \cos z - i \sin z \\ e^{iz} = \cos z + i \sin z \end{array} \right]$

$f(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix} \cdot e^{-y} - e^{-ix} e^{y}}{2i} =$

$= \frac{(\cos x + i \sin x) e^{-y} - (\cos x - i \sin x) e^{y}}{2i} = \frac{\cos x (e^{-y} - e^{y}) + i \sin (e^{-y} + e^{y})}{2i} \cdot \frac{i}{i} =$

$= \sin x \, ch y + i \cos x \, sh y$

FONTOS $\sin z \in \mathbb{C}$ NEM KORL, PERIODIKUS

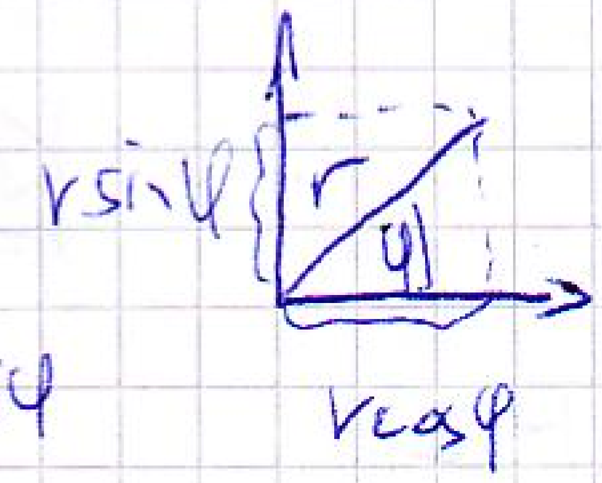
2π periodikus $\sin(z+2\pi) = \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} = \frac{e^{i2\pi} e^{iz} - e^{-i2\pi} e^{-iz}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} = \sin z$

$\left[e^{i2\pi} = \cos 2\pi + i \sin 2\pi = 1 \right]$

⑤ házi: $\cos z = \cos x \operatorname{ch} y - i \sin x \operatorname{sh} y$

⑥ $f(z) = \ln z = w$

$e^{w} = e^{\ln z} = z$



$x+iy = r\cos\varphi + r i \sin\varphi = r(\cos\varphi + i\sin\varphi) = r\cdot e^{i\varphi}$

$z = |z|\cdot e^{i \operatorname{arc} z} = |z|\cdot e^{i\varphi}$

$\varphi = \operatorname{arc} z$
$r = |z|$ sugár

$\ln z = \ln |z| e^{i} = \ln|z| + \ln e^{i \operatorname{arc} z} = \underbrace{\ln|z|}_{\operatorname{Re}\{\}} + \underbrace{i \operatorname{arc} z}_{\operatorname{Im}\{\}}$

$\ln_k z = \ln|z| + i(\operatorname{arc} z + 2k\pi)$ $\Rightarrow$ több fv.

végtelen sok értékű

$e^{\ln_k z} = z \quad (\forall k)$

$k = 0$ esetén FŐÉRTÉK: $\ln_z = \ln_0 z$

Tehát ln fv. 2π periodikus fv.

⑦ $f(z) = \dfrac{z+1}{z-1} = \dfrac{x+iy+1}{(x-1)+iy} = \dfrac{(x+1)+iy}{(x-1)+iy}\cdot\dfrac{(x-1)-iy}{(x-1)-iy} =$

$= \dfrac{x^2-1+y^2+i(-y(x+1)+y(x-1))}{(x-1)^2+y^2} = \underbrace{\dfrac{x^2-1+y^2}{(x-1)^2+y^2}}_{\operatorname{Re}} + i\underbrace{\dfrac{-2y}{(x-1)^2+y^2}}_{\operatorname{Im}}$

$\boxed{\text{II}}$ $\sin z = \sin x \operatorname{ch} y + i \cos x \operatorname{sh} y$

$\operatorname{ch} y = \dfrac{e^y+e^{-y}}{2} > 0$

$\sin z = 0 \Leftrightarrow$ $\sin x \operatorname{ch} y = 0 \Rightarrow \sin x = 0 \Rightarrow x = k\pi$
$\cos x \operatorname{sh} y = 0 \Rightarrow \operatorname{sh} y = 0 \Rightarrow y = 0$

cos soha 0, ha $x = k\pi$ $\quad \operatorname{sh} y = \dfrac{e^y - e^{-y}}{2} = 0$

$\Rightarrow e^y = e^{-y} \Leftrightarrow \boxed{y = -y} \Rightarrow 0 = y$

$\sin|z| = 0 \Leftrightarrow z = k\pi$ (ugyanaz, mint valós esetben)

$\boxed{III}$

① $\cos(-i)$ $\quad x=0$, $y=-1$

$$\cos(z) = \cos x \, ch y - i \sin x \, sh y$$

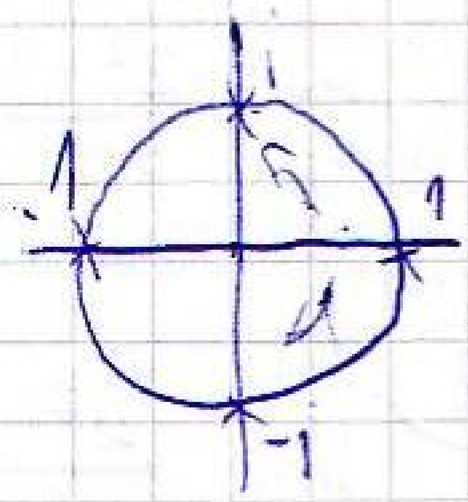
$$\cos(-i) = \underbrace{\cos 0}_{1} \, ch(-1) - i \underbrace{\sin(0)}_{0} \, sh(-1) = ch(-1)$$

② $sh\left(1+i\frac{\pi}{2}\right) = ?$

$$sh\, z = \frac{e^{z} - e^{-z}}{2}$$

$$sh\left(1+i\frac{\pi}{2}\right) = \frac{e^{1+i\frac{\pi}{2}} - e^{-1-i\frac{\pi}{2}}}{2} = \frac{e \cdot e^{i\frac{\pi}{2}} - e^{-1} e^{-i\frac{\pi}{2}}}{2} = \underline{\underline{\frac{i\left(e+\frac{1}{e}\right)}{2}}}$$

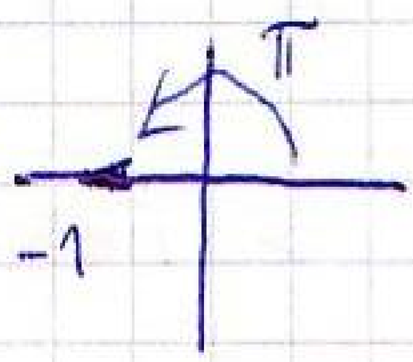
$$\left(\begin{array}{l} e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i \\ e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i \end{array}\right)$$



③ $\ln(-1)$ $\quad$ valós esetben nincs értelme!

$x=-1$
$y=0$

$$\ln z = \ln|z| + i \arc z \qquad z \neq$$



$$-1 = r \cdot e^{i\varphi}$$

$r=1$
$\varphi = \pi$

$$\ln(-1) = \underbrace{\ln 1}_{0} + i(\pi + 2k\pi) = i(2k+1)\pi$$

(4) $Re = ?$, $Im = ?$

$r = 1$

$\varphi = \frac{\pi}{2}$

$$i^i = ? = e^{\ln i^i} = e^{i \ln i} \Rightarrow \ln i = \ln 1 + i\left(\frac{\pi}{2} + 2k\pi\right) = i\left(\frac{\pi}{2} + 2k\pi\right)$$

$$e^{i \ln i} = e^{-\left(\frac{\pi}{2} + 2k\pi\right)} = i^i$$

Határérték számítás

(1) def (ugyanaz mint valósban)

$$\lim_{z \to i} \frac{z^2+1}{z^6+1} \qquad \left|\frac{0}{0}\right| \Rightarrow \text{L'Hospital}$$

$$i^6 = i^2 \cdot \underbrace{i^4}_{1} = -1$$

$$\lim_{z \to i} \frac{z^2+1}{z^6+1} = \lim_{z \to i} \frac{2z}{6z^5} = \lim_{z \to i} \frac{1}{3\underbrace{z^4}_{1}} = \underline{\underline{\frac{1}{3}}}$$

(2) $$\lim_{z \to 0} \frac{e^z - 1}{z} = \lim_{z \to 0} \frac{e^z}{1} = \underline{\underline{1}}$$

(3) Terjesszük folytonossá 0-ban a köv. fv-ket

a) $$f(z) = \frac{z \cdot Re\{z\}}{|z|}$$

$$\lim_{z \to 0} f(z) = \lim \frac{z \cdot Re\{z\}}{|z|} = \qquad z = re^{i\varphi}$$

ha $z \to 0 \Rightarrow r \to 0$

$$= \lim_{r \to 0} \frac{re^{i\varphi} \cdot r\cos\varphi}{r} = \lim_{r \to 0} r \underbrace{e^{i\varphi}\cos\varphi}_{\text{korl}} = 0$$

$$g(z) = \begin{cases} f(z) & z \neq 0 \\ 0 & z = 0 \end{cases} \qquad \text{folyt} \ldots$$

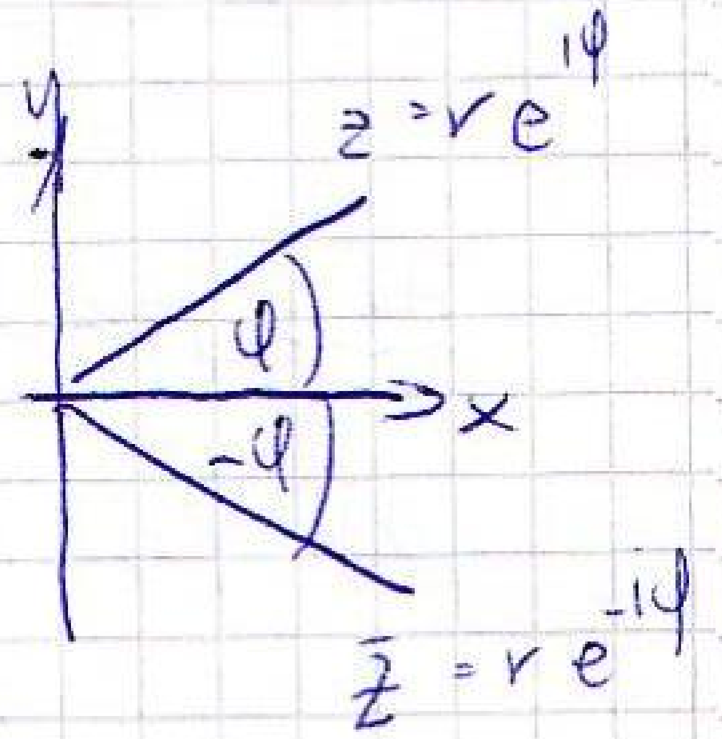
b) $f(z) = e^{\frac{1}{|z|}}$

$$\lim_{z\to 0} e^{\frac{1}{|z|}} = \lim_{r\to 0^+} e^{\frac{1}{r}} = +\infty$$

origóban nem lehet folyt.-sá tenni

c) $f(z) = \frac{z}{|z|}$

$$\lim_{z\to 0} \frac{z}{|z|} = \lim_{r\to 0} \frac{r \cdot e^{i\varphi}}{r} = e^{i\varphi}$$ függ a φ-től



$\Rightarrow \nexists$ HÉ

d)

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{r\to 0} \frac{r\, e^{-i\varphi}}{r\, e^{i\varphi}} = e^{-2i\varphi}$$ függ φ-től ?

$\nexists$ a HÉ

Regularitás

$f(z)$ z_0-ban reguláris, ha z_0 valamely környezetében diffható ($K(z_0)$-ban)
(mindegyik Taylor sorba fejthető)

$f(z) = u(x,y) + i v(x,y)$ reguláris, ha $\boxed{\begin{matrix} u'_x = v'_y \\ u'_y = -v'_x \end{matrix}}$ Cauchy-Riemann diff. egyenletrsz.

$$f'(z) = u'_x + i v'_x = v'_y + i v'_x = v'_y - i u'_y = u'_x - i u'_y$$

Ha teljesül

$u''_{xx} = v''_{yx}$ $\qquad v''_{xy} = v''_{yx}$ Young-Schwarz tétel

$u''_{yy} = -v''_{xy}$ $\qquad \boxed{u''_{xx} + u''_{yy} = 0} \Rightarrow$

$\nabla = \left(\partial_x, \partial_y\right)$

$\nabla^2 = \Delta$ Laplace-operátor

$u''_{xx} + u''_{yy} = 0$

$$\boxed{\begin{array}{c} \Delta u = \phi \\ \Delta v = \phi \end{array}}$$ Laplace egyenlet

$\approx$ legegyszerűbb hullámegyenlet

Hol regulárisak a köv. fv-ek?

a) $f(z) = z \cdot |z| = \dots$ $u = ?$ $v = ?$

$= (x+iy)\sqrt{x^2+y^2} = \underbrace{x\sqrt{x^2+y^2}}_{u(x,y)} + i\underbrace{y\sqrt{x^2+y^2}}_{v(x,y)}$

$u'_x = \sqrt{x^2+y^2} + x \cdot \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}} 2x = \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$

$u'_y = x\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}} 2y = \frac{xy}{\sqrt{x^2+y^2}}$

$v'_x = \frac{xy}{\sqrt{x^2+y^2}}$

$v'_y = \frac{2y^2+x^2}{\sqrt{x^2+y^2}}$

$u'_x \stackrel{?}{=} v'_y$

$2x^2+y^2 = 2y^2+x^2$

$x^2 = y^2 \Rightarrow$ $x = -y$ vagy $x = y$

$u'_y = -v'_x$

$x \cdot y = -x \cdot y$

$2xy = 0$

$\Rightarrow$ $x = 0$ vagy $y = 0$

$x = y = 0$-ban reguláris, máshol nem

b) $f(z) = \text{Re } z = x$

$f(z) = \ln z$ sehol sem reguláris

$u'_x = 1 \quad v'_x = 0$

$u'_y = 0 \quad v'_y = 0$

sehol sem reguláris!

c) $f(z) = e^z = e^x \cdot e^{iy} = e^x(\cos y + i \sin y) = \underbrace{e^x \cos y}_{u(x,y)} + i \underbrace{e^x \sin y}_{v(x,y)}$

$u'_x = e^x \cos y \qquad v'_x = e^x \sin y$

$u'_y = -e^x \sin y \qquad v'_y \; e^x \cos y$

MINDENÜTT REGULÁRIS $\Rightarrow$ ANALITIKUS (cos, sin, ch, sh is az)

d) $f(z) = e^z \cdot |z| = (e^x \cos y + i e^x \sin y)\sqrt{x^2+y^2} =$

$= \underbrace{e^x \sqrt{x^2+y^2} \cos y}_{u(x,y)} + i \underbrace{e^x \sqrt{x^2+y^2} \sin y}_{v(x,y)}$

$u'_x = e^x \sqrt{x^2+y^2} \cos y + e^x \frac{x}{\sqrt{x^2+y^2}} \cos y$

$u'_y = e^x \frac{y}{\sqrt{x^2+y^2}} \cos y - e^x \sqrt{x^2+y^2} \sin y$

$v'_x = e^x \sqrt{x^2+y^2} \sin y + e^x \frac{x}{\sqrt{x^2+y^2}} \sin y$

$v'_y = e^x \frac{y}{\sqrt{x^2+y^2}} \sin y + e^x \sqrt{x^2+y^2} \cos y$

I. $y \sin y = x \cdot \cos y$

II. $y \cos y = -x \sin y$ → feltéve $\neq 0$

$\frac{I.}{II.} \quad \text{tg } y = -\text{ctg } y$

$\text{tg } y = z$

$z = -\frac{1}{z} \quad (z \neq 0)$

$z^2 = -1 \; \begin{cases} i \\ -i \end{cases}$

valós

ez valós

$\mathrm{tg}\, y = i \qquad \mathrm{tg}\, y = -i$

tg valós sosem lesz komplex $\Rightarrow$ sehol sem reguláris