2007.05.10. GYAKORLAT

CSÜTÖRTÖK

Jövő gyak PÓT ZH - Nem kell jönni

INTEGRÁLÁS, STB... :)

$$f(z) = u(x,y) + i\,v(x,y)$$

$$G: z = z(t) = x(t) + i \cdot y(t) \qquad \alpha \leq t \leq \beta$$

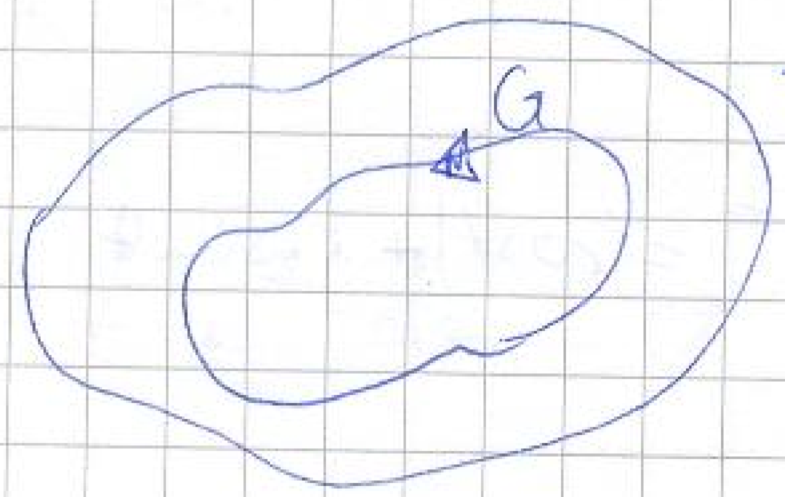
$$\int_{\substack{A \\ G}}^{B} f(z)\,dz = \int_\alpha^\beta \underbrace{f(z(t))}_{u(x(t),\,y(t)) + i \cdot v(x(t),\,y(t))} \overbrace{\dot{z}}^{\dot{x}(t) + i \cdot \dot{y}(t)}\,dt =$$

$$= \underbrace{\int_\alpha^\beta \left(u(x,y) \cdot \dot{x}(t) - v(x,y) \cdot \dot{y}(t)\right) dt}_{Re} +$$

$$i \underbrace{\int_\alpha^\beta \left(u(x,y)\,\dot{y} + v(x,y)\,\dot{x}\right) dt}_{Im}$$

Cauchy - tétel :

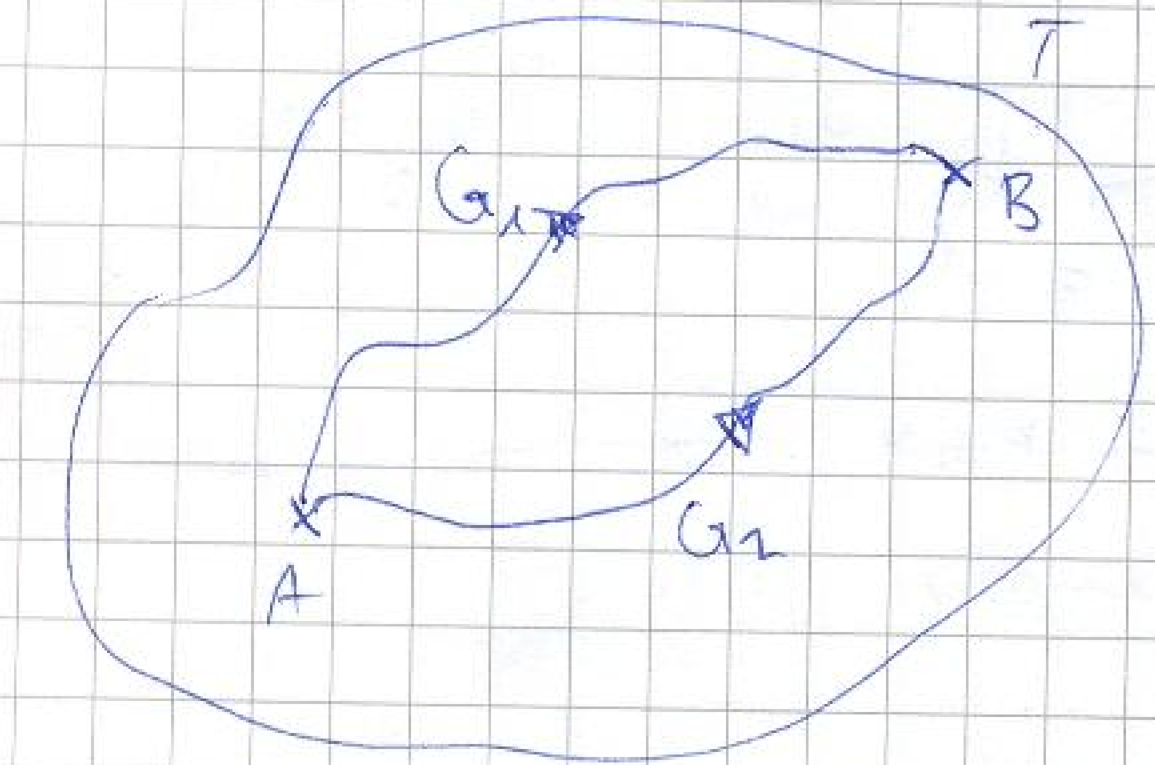
T egyszeresen összefüggő tartomány, G zárt görbe



Ha $f(z)$ reguláris T-ben (C-R feltételek teljesülnek)

$$\Rightarrow \oint_G f(z)\,dz = 0$$

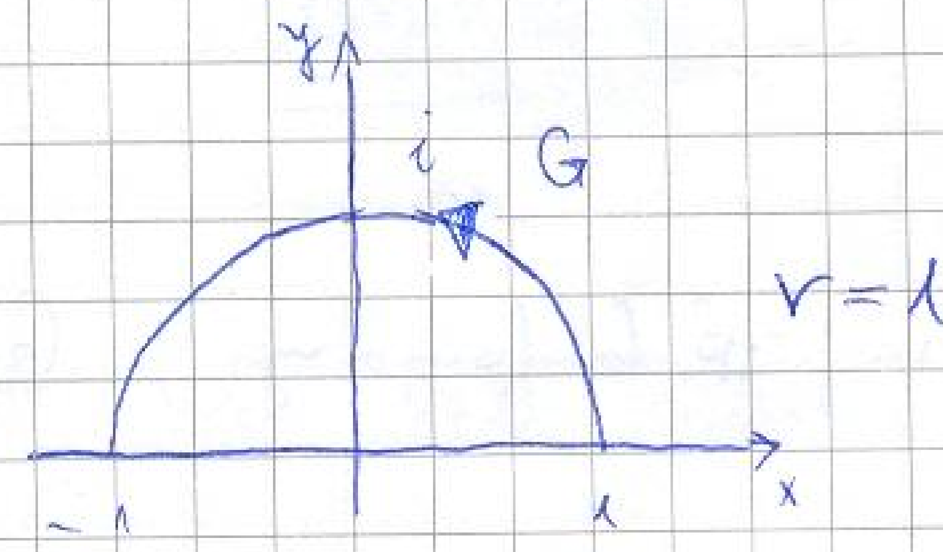
/ regularitás $\overset{?}{\Leftrightarrow}$ potenciálos /



$$\underset{(G1)}{\int_A^B f(z)\,dz} = \underset{(G2)}{\int_A^B f(z)\,dz}$$

Példás: $\boxed{1.}$

$f(z) = |z| \cdot \bar{z}$



$z = r \cdot e^{it}$ $\qquad e^{it} = \cos(t) + i \cdot \sin(t)$

$z(t) = e^{it} \qquad +\pi \geq t \geq 0$

$$\int_G f(z)\,dz = + \int_0^{\pi} f(z)\,dz$$

$$|z(t)| = |e^{it}| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

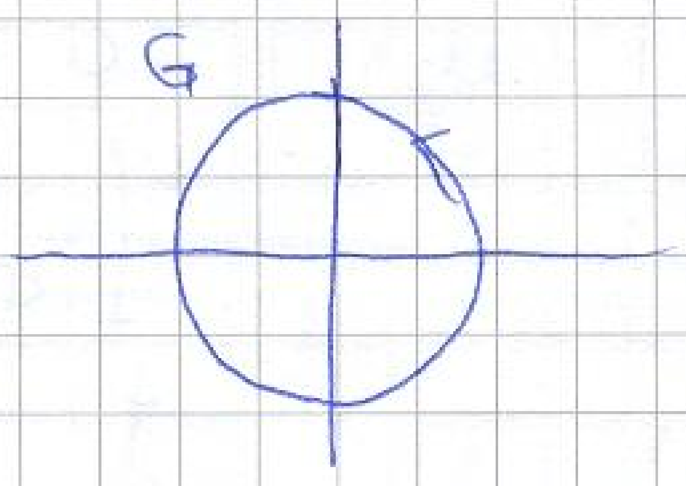
$$\bar{z}(t) = e^{-it}$$

$$\dot{z}(t) = i \cdot e^{it}$$

$$+\int_0^{\pi} e^{-it}\, i\, e^{it}\, dt = +i\,[t]_0^{\pi} = \underline{\underline{+i \cdot \pi}}$$

$\boxed{2}$

$$f(z) = z \cdot e^z$$



$$G: \ |z| = 1$$

$$\oint_G z \cdot e^z \, dz = \emptyset$$

Newton-Leibnitz FORMULA ugyanúgy használható

Biz:

$$G: \ z(t) = e^{it} \qquad 0 \le t \le 2\pi$$

$$f(z(t)) = e^{it} \cdot e^{e^{it}}$$

$$\dot{z}(t) = i \cdot e^{it}$$

$$\int_0^{2\pi} e^{it} \cdot e^{e^{it}} \cdot i \cdot e^{it} \, dt =$$

* * *

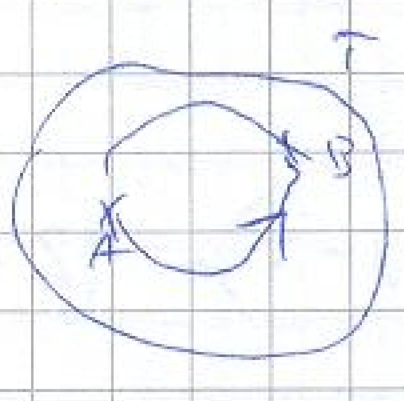
$f(z)$ reguláris T-n

$\exists$ primitív fv

$\Downarrow$

$\oint f(z)\, dz = \emptyset$

$$\oint f(z)\, dz$$



$$\int_A^B f(z)\, dz = F(B) - F(A)$$

* * *

ezért primit fv-nyel:

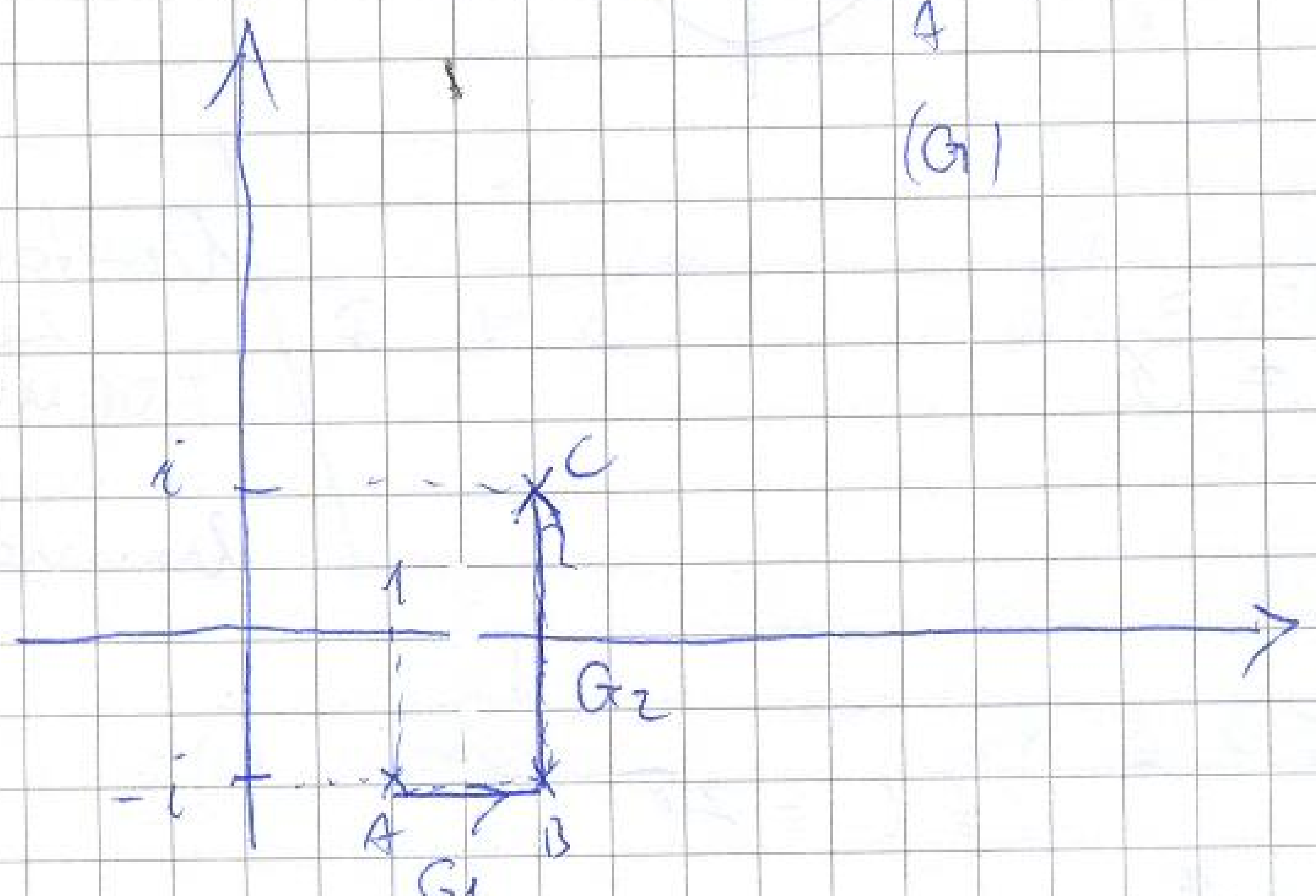
$$\int z \cdot e^z \, dz = z e^z - \int e^z = z \cdot e^z - e^z \quad \text{primitív fv}$$

$\exists$ primitív fv $\Longleftrightarrow$ $\oint f(z)\, dz = \emptyset$!!!

$\boxed{3}$

$f(z) = 3z^2 + 2z$ $\qquad \int\limits_{\substack{A \\ (G)}}^{C} f(z)\,dz = ?$

G_1 :



$\cancel{(1-i)t + (2-i)(1-t)}$

AB: $G_1 = (1-i)(1-t) + (2-i)t$: paraméterezése

megj: A-ból B-be egyenes mentén:

$A(1-t) + Bt \qquad 0 \le t \le 1$

$G_1: z_1(t) = (1-i)(1-t) + (2-i)t = t + 1 - i$

$\cancel{1} - \cancel{t} - i + \cancel{it} + \cancel{2t} - \cancel{it}$

$G_2: z_2(t) = (2-i)(1-s) + (2+i)s = 2i \cdot s + 2 - i \quad 0 \le s \le 1$

$2 - \cancel{2s} - i + \cancel{is} + \cancel{2s} + \cancel{is}$

$\int\limits_{\substack{A \\ (G)}}^{C} f(z)\,dz = \int\limits_{\substack{A \\ (G_1)}}^{B} f(z)\,dz + \int\limits_{\substack{B \\ (G_2)}}^{C} f(z)\,dz =$

↪ folyt.

$$\int_0^1 \left[3\left(t+1-i\right)^2+2\left(t+1-i\right)\right]\cdot 1\,dt+\int_0^1\left[3\left(2is+2-i\right)^2+2\left(2is+2i\right)\right]2i\,ds$$

$$\int_0^1 \left[3(t+1-i)^2+2(t+1-i)\right]\cdot 1\,dt+\int_0^1\left[3(2is+2-i)^2+2(2is+2-i)\right]2i\,ds$$

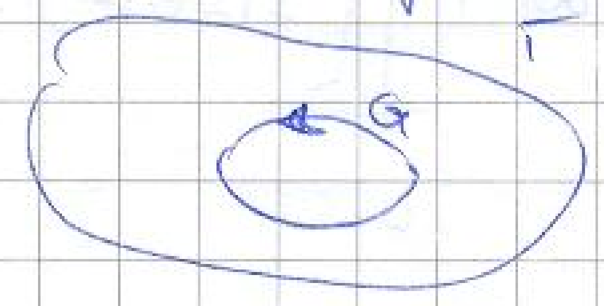
$3((t+1)^2-1,+2i(t+i))$

$$\left[(t+1)^3+(-1-5i)t+(2-6i)\frac{t^2}{2}\right]_0^1$$

ifb

Cauchy-féle integrál formula

T 1x-en összefüggő, f reguláris T-ben



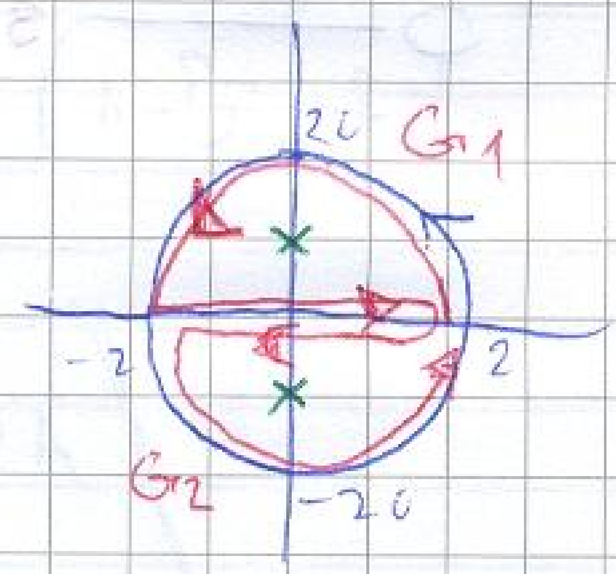
(1)
$$f(z) = \frac{1}{2\pi i} \oint_G \frac{f(\xi)}{\xi - z}\, d\xi$$

(2)
$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_G \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi \qquad n = 1, 2, \dots$$

példa 1

$\boxed{4}$

$$\oint \frac{dz}{1+z^2} \qquad G : |z| = 2$$



$\left.\begin{matrix} z = i \\ z = -i \end{matrix}\right\}$ szingularitások $\qquad G : G_1 \cup G_2$

$$\oint \frac{dz}{1+z^2} = \oint \frac{dz}{(z-i)(z+i)} = \oint_{G_1} \bigcirc + \oint_{G_2} \bigcirc =$$

reguláris G_1 belsejében $\qquad$ reguláris G_2 belsejében

$$= \underbrace{\oint_{G_1} \frac{\frac{1}{z+i}}{z-i}\, dx}_{\underbrace{\frac{1}{2\pi i} \cdot \left.\frac{1}{z+i}\right|_{z=i}}_{-\frac{1}{4\pi}}} + \underbrace{\oint_{G_2} \frac{\frac{1}{z-i}}{z+i}\, dz}_{\underbrace{\frac{1}{2\pi i} \cdot \left.\frac{1}{z-i}\right|_{z=-i}}_{\frac{1}{4\pi}}} =$$

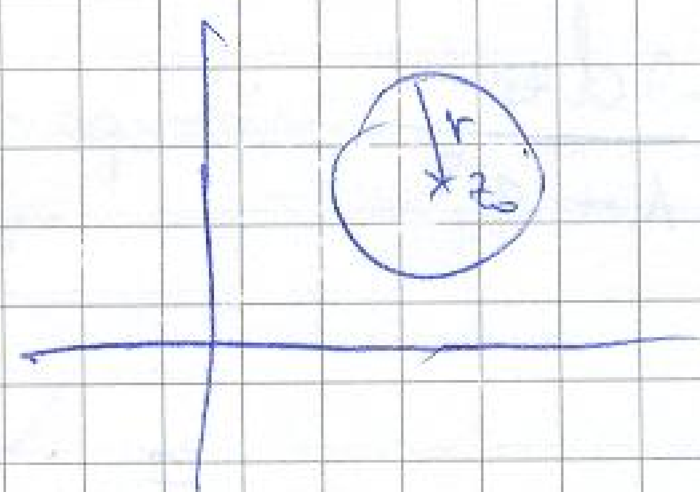
$$= -\frac{1}{4\pi} + \frac{1}{4\pi} = \emptyset$$

példa 2 -re piros

$\boxed{5}$

$$\oint \frac{dz}{z(z^2-1)^3} \qquad G \quad |z+1| = \frac{3}{4}$$

Alt $\quad |z-z_0| = r$

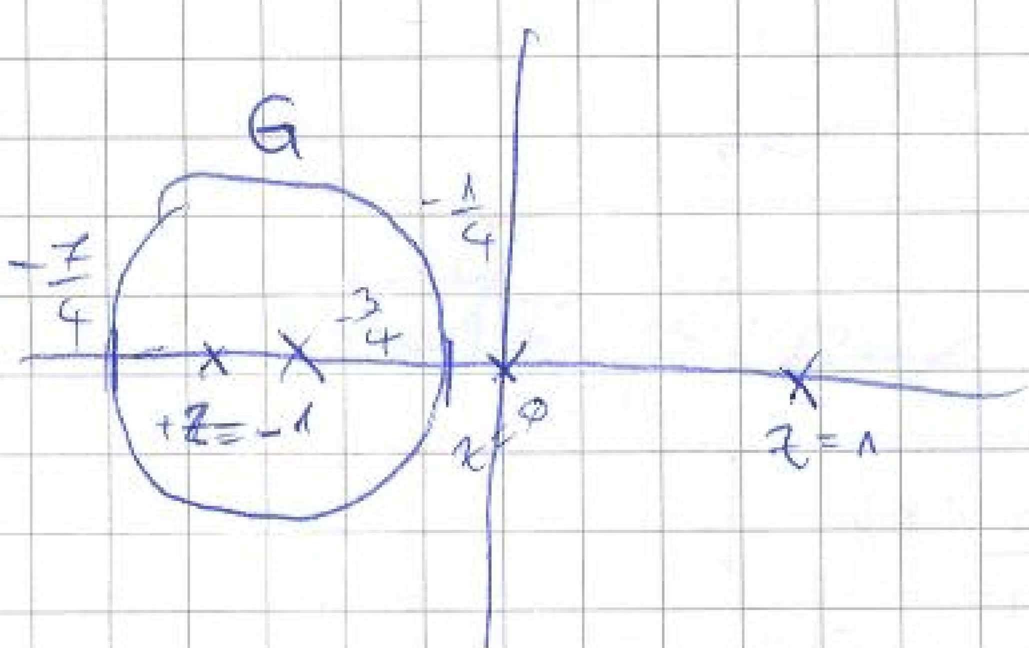


Szingularitások:

$z = 0$

$z = 1$

$z = -1$



reguláris G-n belül

$$\oint \frac{dz}{z(z-1)^3(z+1)^3} = \oint \frac{\frac{1}{z(z-1)^3}}{(z+1)^3}\, dz = =$$

$$= \frac{1}{2\pi i} \left. \frac{1}{z(z-1)} \right|_{z=-1} = \underline{\underline{\frac{1}{4\pi}}}$$

* $n=2$ esetre

$$\Rightarrow f(z) = \frac{1}{z(z-1)^3}$$

$$f'(z) = \left(z^{-1} \cdot (z-1)^{-3}\right)' = -\frac{1}{z^2}\frac{1}{(z-1)^3} + \frac{1}{z}(-3)\frac{1}{(z-1)^4}$$

$$f''(z) = -\frac{2}{z^3}\frac{1}{(z-1)^3} + \frac{3}{z^2}\cdot\frac{1}{(z-1)^4} + \frac{3}{z^2}\frac{1}{(z-1)^4} + \frac{12}{z}\frac{1}{(z-1)^5}$$

$$= \underline{\underline{\frac{2\pi i}{2!} f''(-1)}} \checkmark$$

SOROK

konvergencia tartomány

$$f(z) = \sum_k c_k (z - z_0)^k$$

① $k = 0, 1, 2, 3 \ldots$ stb. : Taylor-sor

lehetséges konv. tartomány:

- csak z_0-ban
- $|z - z_0| < R$ → konvergencia sugár: $\frac{1}{R} = \limsup \sqrt[n]{|c_n|} = \limsup \left|\frac{c_{n+1}}{c_n}\right|$
- $\mathbb{C}$-n (analitikus fv-ek)

DUÁLIS

② $k = -1, -2, -3, \ldots$ stb : NTL - sor

Negatív tagú Laurent-sor

- z_0 kivételével bárhol
- sehol nem konvergens
- $|z - z_0| > R$

③ $k \in \mathbb{Z}$: L-sor

- sehol nem
- $r < |z - z_0| < R$
- $\mathbb{C} / \{z_0\}$

Példák

$\boxed{6}$

$$f(z) = \frac{1}{(z-2)(z-1)} \qquad z_0 = \emptyset \text{ körül}$$

$$\left.\begin{matrix} z = 2 \\ z = 1 \end{matrix}\right\} \text{ szinguláris pontok}$$

a) $|z| < 1$

b) $1 < |z| < 2$

c) $|z| > 2$

Sorfejtés:

a) $$\frac{1}{(z-2)(z-1)} = \frac{A}{z-2} + \frac{B}{z-1} = \frac{A(z-1) + B(z-2)}{\cancel{(z-2)(z-1)}}$$

$z = 1$ esetén $\quad B = -1$

$z = 2 \quad$ -"- $\quad A = 1$

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x} \quad |x| < 1$$

ha $|z| < 1$ $\qquad \left|\frac{z}{2}\right| < 1$

$$\frac{1}{z-2} = -\frac{1}{2} \frac{1}{1-\frac{z}{2}} = -\frac{1}{2} \sum_{k=0}^{\infty} \left(+\frac{z}{2}\right)^k =$$

$$= -\frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} \cdot z^k$$

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{k=0}^{\infty} z^k$$

$$\frac{1}{(z-2)(z-1)} = -\frac{1}{2}\underbrace{\sum_{k=0}^{\infty} \frac{1}{2^k} z^k + \sum_{k=0}^{\infty} z^k}_{\sum_{k=0}^{\infty} \frac{1}{1-2^k} z^k}$$

b) $1 < |z| < 2$

$$\frac{1}{z-2} = -\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k \rightarrow -$$

$\frac{z}{2} < 1$

$$\frac{1}{z-1} = \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k = \sum_{k=1}^{\infty}\left(\frac{1}{z}\right)^k = \sum_{k=-1}^{-\infty} z^k$$

$\frac{1}{z} < 1$

$$-\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k - \sum_{k=-1}^{-\infty} z^k$$

c) $|z| > 2$

$$\frac{1}{z-2} = \frac{1}{z} \frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k = \sum_{k=1}^{\infty} \frac{2^{k-1}}{z^k} =$$

$$\left|\frac{2}{z}\right| < 1 \qquad = \sum_{k=-1}^{-\infty} \frac{z^k}{2^{k-1}}$$

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k = \sum_{k=-1}^{-\infty} z^k$$

$$\left|\frac{1}{z}\right| < 1$$

KONZULTÁCIÓ!