Let $\mathbf{S} = (S, \wedge, m)$ be a finite algebra so that $(S, \wedge)$ is a semilattice, and m is a majority operation on S. For subsets $S_1 \subseteq S_2$ of S we say that S_1 absorbs S_2 ($S_1 \lhd_m S_2$) if $m(x, y, z) \in S_1$ whenever x, y, and z are both in S_2, and at least two of them are in S_1. If furthermore $x \wedge y \in S_1$ whenever x and y are in S_2, and one of them is in S_1, then we call S_1 and *ideal* of S_2, denoted by $S_1 \lhd S_2$. We call an operation of S *m-generated* if it is a term of (S, m). We recursively define for natural z the 3^z-ary term $m^{(z)}$ with $m^{(1)} = m$, and

$$m^{(z)}(x_1, \dots, x_{3^z}) :=$$
$$m(m^{(z-1)}(x_1, \dots, x_{3^{z-1}}), m^{(z-1)}(x_{3^{z-1}+1}, \dots, x_{2*3^{z-1}}), m^{(z-1)}(x_{2*3^{z-1}+1}, \dots, x_{3^z}))$$

for $z > 1$. So an m-generated operation is just a polymer of an $m^{(z)}$ for a large enough z.

We sort linear identities to four groups:

- *symmetry-type identities*: identities of the form $t_1(x_{i_1}, \dots, x_{i_n}) = t_2(x_{j_1}, \dots, x_{j_l})$, where the sets $\{x_{i_1}, \dots, x_{i_n}\}$ and $\{x_{j_1}, \dots, x_{j_l}\}$ coincide,
- *irrelevance-type identities*: identities of the form $t_1(x_{i_1}, \dots, x_{i_n}) = t_2(x_{j_1}, \dots, x_{j_l})$, where the sets $\{x_{i_1}, \dots, x_{i_n}\}$ and $\{x_{j_1}, \dots, x_{j_l}\}$ differ,
- *restricted majority-type identities*: identities of the form $t(x_{i_1}, \dots, x_{i_n}) = x$ with $|\{k : x_{i_k} \neq x\}| > |\{x_{i_k} : x_{i_k} \neq x\}|$,
- *unrestricted majority-type identities*: identities of the form $t(x_{i_1}, \dots, x_{i_n}) = x$ with $|\{k : x_{i_k} \neq x\}| = |\{x_{i_k} : x_{i_k} \neq x\}|$.

Definition 1. For an n-ary term t, we call a set $R \subseteq \{1, 2, \dots, n\}$ *big for* t if t satisfies a majority-type identity $t(x_{i_1}, \dots, x_{i_n}) = x$ with $\{r : x_{i_r} = x\} = R$.

Proposition 2. *If t is an n-ary lattice term, and R is big for t, then t satisfies* any *majority-type identity $t(x_{i_1}, \dots, x_{i_n}) = x$ with $\{r : x_{i_r} = x\} = R$.*

Proof. PROOF NEEDED. EASIER FOR DISTRIBUTIVE LATTICE TERMS, BUT NOT HARD FOR GENERAL EITHER. □

Therefore, a lattice term satisfies a restricted majority identity iff it satisfies the corresponding unrestricted identity.

For any n-ary term t we introduce the following set:

$$\mathcal{M}_t^{(1)} := \{u \in \text{Term(maj.)} : u(y_1, \dots, y_n) = y$$
is an identity in the majority-generated variety whenever $\{r : y_r = y\}$ is big for $t\}$,

and then the $\mathbf{S}$-term
$$\bar{t}^{(1)} := \bigwedge_{u \in \mathcal{M}_t} u.$$

Even though $\mathcal{M}_t$ is infinite, $\bar{t}^{(1)}$ must be a meet of one of its finite subsets (as S is finite), therefore indeed $\bar{t}^{(1)} \in \text{Term}(\mathbf{S})$.

Lemma 3. *For all $x_1, \dots, x_n \in S$,*

$$\bar{t}^{(1)}(x_1, \dots, x_n) = \bigwedge \{s \in S : \forall S' \lhd_m [x_1, \dots, x_n]_m : (\{r : x_r \in S'\} \textit{ is big for } t \Rightarrow s \in S')\},$$

where $[x_1,\dots,x_n]_m$ *denotes the subset of* S m*-generated by* $x_1,\dots,x_m$*, i.e. the set*

$$\{f(x_1,\dots,x_n) : f\in \mathrm{Term}_n(\mathbf{S}),\ f \textit{ is } m\textit{-generated}\}.$$

Proof. For subsets $I_1,\dots,I_h$ of $\{1,2,\dots,n\}$, we introduce the notation

$$\mathcal{M}^{(1)}_{I_1,\dots,I_h} := \{u\in \mathrm{Term(maj.)} :\ u(y_1,\dots,y_n)=y$$
is an identity in the majority-generated variety whenever $\{r : y_r=y\}\in\{I_1,\dots,I_h\}\}$.

Furthermore,

$$\mathcal{M}^{(1)}_{I_1,\dots,I_h}(x_1,\dots,x_k) := \{u(x_1,\dots,x_k) : u\in\mathcal{M}_{I_1,\dots,I_h}\}$$

and

$$\mathcal{M}^{(1)}_t(x_1,\dots,x_k) := \{u(x_1,\dots,x_k) : u\in\mathcal{M}_t\}.$$

Thus $\mathcal{M}^{(1)}_{I_1,I_2}=\mathcal{M}^{(1)}_{I_1}\cap\mathcal{M}^{(1)}_{I_2}$, but $\mathcal{M}^{(1)}_{I_1,I_2}(x_1,\dots,x_n)$ and $\mathcal{M}^{(1)}_{I_1}(x_1,\dots,x_n)\cap\mathcal{M}^{(1)}_{I_2}(x_1,\dots,x_n)$ may differ.

Suppose that $u\in\mathcal{M}^{(1)}_t$, and $S'\lhd_m [x_1,\dots,x_n]_m$ is such that $R:=\{r : x_r\in S'\}$ is big for t. We claim that $u(x_1,\dots,x_n)\in S'$. This is obviously true if u is a projection, let us proceed with induction on the complexity of u: suppose that $u=m(u_1,u_2,u_3)$ with u_1,u_2,u_3 being m-generated $\mathbf{S}$-terms. As $u\in\mathcal{R}$, at least two of the terms u_1, u_2, and u_3 must also be in $\mathcal{R}$. This means by the inductive hypothesis that at least two of $u_1(x_1,\dots,x_n)$, $u_2(x_1,\dots,x_n)$, and $u_3(x_1,\dots,x_n)$ are in S'. All three are obviously in $[x_1,\dots,x_n]_m$. Thus, $u(x_1,\dots,x_n)$ is in S'.

Consequentially,

$$\bar{t}^{(1)}(x_1,\dots,x_n)\geq \bigwedge\{s\in S : \forall S'\lhd_m [x_1,\dots,x_n]_m : (\{r : x_r\in S'\}\text{ is big for } t\ \Rightarrow\ s\in S')\},$$

because the left side is the meet of all elements of S of the form $u(x_1,\dots,x_n)$ with $u\in\mathcal{M}^{(1)}_t$, and all such elements are in the set of which the right side is the meet of. We need to prove the converse inequality. To do that, it is enough to show that for any $s\in S$ satisfying

$$\forall S'\lhd_m [x_1,\dots,x_n]_m : (\{r : x_r\in S'\}\text{ is big for } t\ \Rightarrow\ s\in S'),$$

there is an $u\in\mathcal{M}^{(1)}_t$ with $u(x_1,\dots,x_n)=s$.

Suppose that $J_1,J_2\subseteq\{1,2,\dots,n\}$ are big for t. Define T_1 and T_2 as the smallest absorbing subset of $[x_1,\dots,x_n]_m$ containing $\{x_i : i\in J_1\}$ and $\{x_i : i\in J_2\}$, respectively. It is easy to see that for a large enough z

$$T_1=\{m^{(z)}(a_{0\dots0},\dots,a_{2\dots2} : a_{e_1e_2\dots e_z}\in\{x_1,\dots,x_n\},$$
$$\forall(f_1,\dots,f_z)\in\{0,1\}^z : a_{f_1f_2\dots f_z}\in\{x_i : i\in J_1\}\},$$

and

$$T_2=\{m^{(z)}(a_{0\dots0},\dots,a_{2\dots2} : a_{e_1e_2\dots e_z}\in\{x_1,\dots,x_n\},$$
$$\forall(f_1,\dots,f_z)\in\{0,1\}^z : a_{f_1f_2\dots f_z}\in\{x_i : i\in J_2\}\}.$$

This means that there are indexes $b_{0\dots0},\dots,b_{2\dots2},c_{0\dots0},\dots,c_{2\dots2}\in\{1,\dots,n\}$ such that $x_{b_{f_1f_2\cdots f_z}}\in J_1$ and $x_{c_{f_1f_2\cdots f_z}}\in J_2$ for any $(f_1,\dots,f_z)\in\{0,1\}^z$, and

$$s=m^{(z)}(x_{b_{0\dots0}},\dots,x_{b_{2\dots2}})=m^{(z)}(x_{c_{0\dots0}},\dots,x_{c_{2\dots2}})$$

.

Notice that there is an $1 \leq i^* \leq n$ such that $i^* \in J_1 \cap J_2$. Take the term

$$v_{J_1,J_2}(y_1,\ldots,y_n) := m(m^{(z)}(y_{b_{0\ldots0}},\ldots,y_{b_{2\ldots2}}), m^{(z)}(y_{c_{0\ldots0}},\ldots,y_{c_{2\ldots2}}), y_{i^*}).$$

Check that $v_{J_1,J_2} \in \mathcal{M}^{(1)}_{J_1,J_2}$, and $v_{J_1,J_2}(x_1,\ldots,x_n) = s$. Hence, $s \in \mathcal{M}^{(1)}_{J_1,J_2}(x_1,\ldots,x_n)$.

Suppose that $I_1,\ldots,I_h$ are the subsets of $\{1,2,\ldots,n\}$ that are big for t. Assume that $s \notin \mathcal{M}^{(1)}_t = \mathcal{M}^{(1)}_{I_1,\ldots,I_h}$. There is a minimal k such that there is a k-element set $\{J_1,\ldots,J_k\} \subseteq \{I_1,\ldots,I_h\}$ with $s \notin \mathcal{M}^{(1)}_{J_1,\ldots,J_k}$. We have proven that $k > 2$. That means that there is an m-generated k-ary near-unanimity operation m'. By the minimality of k, there are terms $v_1,\ldots,v_k$ such that $v_i \in \mathcal{M}^{(1)}_{J_1,\ldots,J_{i-1},J_{i+1},\ldots,J_k}$ and $v_i(x_1,\ldots,x_n) = s$ for all $1 \leq i \leq k$. But then,

$$v := m'(v_1,\ldots,v_k) \in \mathcal{M}^{(1)}_{J_1,\ldots,J_k},$$

and $v(x_1,\ldots,x_n) = s$, a contradiction. □

Theorem 4. *The mapping $t \mapsto \bar{t}^{(1)}$ on the set of lattice terms preserves all symmetry-type and majority-type identities.*

Proof. The case for unrestricted majority types is immediate from the definition. By Proposition 2, the restricted case follows.

Take an identity $t_1(z_1,\ldots,z_k) = t_2(z'_1,\ldots,z'_l)$, where $\{z_1,\ldots,z_k\} = \{z'_1,\ldots,z'_l\}$ is the n-element set of variables $\{y_1,\ldots,y_n\}$. Note that for any $Y \subseteq \{y_1,y_2,\ldots,y_n\}$, $\{i : z_i \in Y\}$ is big for t_1 if and only if $\{i : z'_i \in Y\}$ is big for t_2. (An easy consequence of Proposition 2.) Putting an element of S x_i into each variable y_i (not necessarily all different), we deduce that for any $S' \subseteq S$, the set $\{r : x_r \in S'\}$ is big for t_1 iff it is big for t_2. Therfore, $\overline{t_1}(x_1,\ldots,x_n) = \overline{t_2}(x_1,\ldots,x_n)$, because by Lemma 3, they are the meet of the same subset of S. □

NOTATIONS HENCEFORTH MAY NOT BE COMPATIBLE WITH THE ONES BEFORE.

For all natural h and lattice term t, we will define an **S**-term $\bar{t}^{(h)}$, a set $\mathcal{L}^{(h)}_t$ and a set $\mathcal{M}^{(h)}_t$ of **S**-terms so that the following hold:

1. The mapping $t \mapsto \bar{t}^{(h)}$ preserves all the symmetry-type identities and all linear identities where on one side there is an at most h-ary term (MAYBE HAVE A NAME FOR THOSE IDENTITIES),
2. If t at most h-ary, then $\bar{t}^{(h)} = \bar{t}^{(h-1)}$,
3. Suppose that t satisfies exactly l identities of the type

   $$t(\underline{v}_i) = s_i(x_1,\ldots,x_h),$$

   (YES, HAVE A NAME FOR THOSE) then $\mathcal{L}^{(h)}_t = \{u_{i,j} : 1 \leq i < j \leq l\}$, where $u_{i,j}$ is an **S**-term satisfying

   $$u_{i,j}(\underline{v}_i) = \overline{s_i}^{(h-1)}(x_1,\ldots,x_h)$$

   and

   $$u_{i,j}(\underline{v}_j) = \overline{s_j}^{(h-1)}(x_1,\ldots,x_h),$$

(4) $\mathcal{M}_t^{(h)}$ is the set of all **S**-terms that can be represented by a binary-ternary tree whose leaves are some $u_{i,j}$ with $\{u_{1,i}, \ldots, u_{i-1,i}, u_{i,i+1}, \ldots, u_{i,l}\}$ being a dominant set of leaves for all $1 \leq i \leq l$ (YOU NEED SOME DEFINITIONS FOR THIS),

(5) For all h,

$$\bar{t}^{(h)} = \bigwedge_{u \in \mathcal{M}_t^{(h)}} u.$$

We will define $\bar{t}^{(h)}$, $\mathcal{L}_t^{(h)}$ and $\mathcal{M}_t^{(h)}$ recursively. Notice that by the above conditions, if we define the set $\mathcal{L}_t^{(h)}$, we have the definitions for $\mathcal{M}_t^{(h)}$ and $\bar{t}^{(h)}$ as well (for a given t and h).

Lemma 5. *Suppose that $\bar{t}^{(h')}$, $\mathcal{L}_t^{(h')}$ and $\mathcal{M}_t^{(h')}$ are defined for all t and all $1 \leq h' < h$ satisfying the conditions (1)–(5). Then for all t there is a set $\mathcal{L}_t^{(h)}$ satisfying condition (3).*

Proof. Denote with n the arity of t. We have to define the **S**-terms $u_{i,j}$. Note that this $u_{i,j}$ depends on t and h (and the enumeration of the HAVE THAT NAME identities satisfied by t). We will fix $(i,j) = (1,2)$. In the remainder of the proof, we will denote other things by i and j.

We start with $h = 1$. Suppose that the equalities $t(\underline{v}_1) = x$ and $t(\underline{v}_2) = x$ are satisfied, where $\underline{v}_1 = (\underline{v}_1^{(1)}, \ldots, \underline{v}_1^{(1)})$ and $\underline{v}_2 = (\underline{v}_2^{(1)}, \ldots, \underline{v}_2^{(1)})$ are n-tuples of variables, some of them equaling x (there is x on the right side instead of a unary term because **S** is an idempotent algebra).

Take the set $A_1, A_2 \in \{1, 2, \ldots, n\}$, where $A_1 = \{r : \underline{v}_i^{(r)} = x\}$ and $A_2 = \{r : \underline{v}_j^{(r)} = x\}$. Note that A_1 and A_2 cannot be disjoint, otherwise taking the n-tuple $\underline{w}(\underline{w}_1^{(1)}, \ldots, \underline{w}_1^{(1)})$ with $\underline{w}_1^{(r)} = x$ for $r \in A_1$ and $\underline{w}_1^{(r)} = y$ for $r \notin A_1$ we would get $x = t(\underline{(w)}) = y$. Now take an element $r_0 \in A_1 \cap A_2$, and set $u_{1,2}(x_1, \ldots, x_n) = x_{r_0}$.

Now suppose $h > 1$, and that t satisfies the identities $t(\underline{v}_1) = s_1(x_1, \ldots, x_h)$ and $t(\underline{v}_2) = s_2(x_1, \ldots, x_h)$.

Claim 1. *There is a lattice term t' with $\operatorname{ar} t' = n' \leq n$, and n'-tuples $\underline{v'}_1$ and $\underline{v'}_2$ containing only the variables $x_1, \ldots, x_h$ such that $t'(\underline{v'}_1) = s_1(x_1, \ldots, x_h)$ and $t'(\underline{v'}_2) = s_2(x_1, \ldots, x_h)$ are satisfied.*

PROVE FOR DIST. LAT. TERMS ONLY.

We can assume without loss of generality that

$$t(x_1, \ldots, x_n) = (x_{a_{1,1}} \wedge \cdots \wedge x_{a_{1,p}}) \vee \cdots \vee (x_{a_{q,1}} \wedge \cdots \wedge x_{a_{q,p}}).$$

Take the normal forms $s_1(x_1, \ldots, x_h) = e_1 \vee \cdots \vee e_b$, and $s_2(x_1, \ldots, x_h) = f_1 \vee \cdots \vee f_c$, where all the e_i and f_i are meets of variables.

If we write $\underline{v}_1$ into t, then we get $e_1 \vee \cdots \vee e_b$. It can be checked that this means that $e_1, \ldots, e_b$ are precisely the maximal elements of the set $\{\underline{v}_1^{(a_{1,1})} \wedge \cdots \wedge \underline{v}_1^{(a_{1,p})}, \ldots, \underline{v}_1^{(a_{q,1})} \wedge \cdots \wedge \underline{v}_1^{(a_{q,p})}\}$ (maximality meant in the free distributive lattice). Suppose that $e_1 = \underline{v}_1^{(a_{1,1})} \wedge \cdots \wedge \underline{v}_1^{(a_{1,p})}$.

FINISH (IF NEEDED). THERE IS NO WAY IT WILL BE UNDERSTANDABLE LIKE THIS.

We can assume that $t = t'$, $\underline{v}_1 = \underline{v'}_1$, and $\underline{v}_2 = \underline{v'}_2$, because if find an **S**-term for the identity-pair for t', then that term will be good for the corresponding identity-pair for t. So henceforth, both $\underline{v}_1$ and $\underline{v}_2$ only contains the variables $x_1, \ldots, x_h$.

For $1 \leq i < j \leq h$, we consider the minors of s_1 and $\overline{s}_1^{(h-1)}$ obtained by writing x_i instead of x_j. Denote this by $s_1[j \to i]$ and $\overline{s}_1[j \to i]$. Consider the HAVE THAT NAME identities satisfied by s_1. Each is a consequence of (at least) one of the identities of the type

$$s_1(x_1, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_h) = s_1[j \to i](x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

(provided that s_1 depends on all its variables WHAT IF NOT–THEN YOU ALREADY FOUND YOUR U-1,2 FOR SMALLER H).

□