QUASIORDER LATTICES IN CONGRUENCE MODULAR VARIETIES

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1. Introduction

Quasiorders (which in this paper means reflexive, transitive and compatible binary relations) of a universal algebra are a common generalization of congruences and natural orders for some class of structures, for example, lattices and inverse semigroups. The quasiorders of an algebra $\mathbf{A}$ form a lattice denoted by $\operatorname{Quo}\mathbf{A}$, which contains $\operatorname{Con}\mathbf{A}$ (the congruence lattice) as a sublattice. With the involution $\delta \mapsto \delta^{-1}$, where δ^{-1} is definied by

$$(a,b) \in \delta^{-1} \Leftrightarrow (b,a) \in \delta,$$

$\operatorname{Quo}\mathbf{A}$ becomes an *involutive lattice*.

In [4], my advisor and I studied the relationship between quasiorder and congruence lattices, and proved the following theorems:

Theorem 1.1. *If a locally finite variety is congruence distributive, then it is also quasiorder distributive. If it is congruence modular, then it is also quasiorder modular.*

Theorem 1.2. *If a locally finite variety is congruence meet semidistributive, then no quasiorder lattice of one of its algebras contains a sublattice isomorphic to* $\mathbf{M}_3$*, but the quasiorder lattices do not have to be meet semidistributive.*

Thus, congruence meet semidistributivity behaves differently than modularity and distributivity.

We used directed Jónsson and directed Gumm terms (see [7]) to prove the first theorem. These categorize congruence distributivity and congruence modularity just like the regular Jónsson and Gumm terms. For the positive statement of the second theorem, we used the fact that some elements of tame congruence theory work for quasiorders as well as for congruences.

In this paper, I use the second approach to prove a common generalization of the first theorem. This method also works to prove a generalization of a theorem by Czédli, Horváth and Lipparini: this states that in congruence modular varieties, the intersection of the congruences generated by two tolerances coincides with the congruence generated with the two tolerances. (A tolerance is a compatible symmetric binary relation.) This will also be true if we replace tolerances with reflexive compatible relations and congruences with quasiorders, with the restriction that the variety needs to be locally finite.

2. Preliminaries

For an algebra $\mathbf{A}$, $R(\mathbf{A})$ denotes the set (and lattice) of reflexive compatible binary relations, and $\operatorname{Tol}\mathbf{A}$ the set of tolerances. The latter is a sublattice of the former. It is important to note however that while Quo $\mathbf{A}$ is a subset of $R(\mathbf{A})$ and Con $\mathbf{A}$ is a subset of Tol $\mathbf{A}$, they are generally not sublattices. $R(\mathbf{A})$ can be made into an involutive lattice the same way as Quo $\mathbf{A}$.

For any $\delta \in R(\mathbf{A})$ there correspond two equivalences: $\delta^* := \delta \wedge \delta^{-1}$ and $\delta \vee \delta^{-1}$. I note that some authors use the notation δ^* for the latter instead of the former. There is also a poset that naturally corresponds to δ: the factor of δ by δ^*. (This is a poset with underlying set A/δ^*, with $(u,v) \in \delta/\delta^*$ iff there is $(a,b) \in \delta$ such that $a/\delta^* = u$ and $b/\delta^* = v$.) Obviously, if δ is a quasiorder then δ^* is a congruence.

The *terms* of an algebra are those operations on its underlying set that are in the clone generated by the fundamental operations of the algebra. The *polynomials* are those operations that are in the clone generated by the fundamental operations and the constant operations. Hence, a k-ary operation is polynomial iff there is a term t and elements $c_{k+1}, \ldots, c_n$ of the algebra such that $p(x_1, \ldots, x_k) = t(x_1, \ldots, x_k, c_{k+1}, \ldots, c_n)$. The set of k-ary terms of $\mathbf{A}$ are denoted by $\mathrm{Term}_k\,\mathbf{A}$, the set of k-ary polynomials by $\mathrm{Pol}_k\,\mathbf{A}$.

A ternary operation t is called a *Mal'tsev-operation* if it satisfies

$$t(x,x,y) \approx t(y,x,x) \approx y.$$

A term (polynomial) of an algebra that is a Mal'tsev-operation is called a *Mal'tsev-term* (*Mal'tsev-polynomial*).

Proposition 2.1. [5] *If* $\mathbf{A}$ *has a Mal'tsev-polynomial, then* $R(\mathbf{A}) = \mathrm{Tol}\,\mathbf{A} = \mathrm{Quo}\,\mathbf{A} = \mathrm{Con}\,\mathbf{A}$.

Congruence modularity of a variety is characterized by the following Mal'tsev condition.

Theorem 2.2. [3] *For any variety $\mathcal{V}$, the following are equivalent:*

(1) *the algebras of $\mathcal{V}$ have modular congruence lattices,*
(2) *$\mathcal{V}$ admits* Gumm-terms*, that is, there are ternary terms $p_0, \ldots, p_n, q$ of $\mathcal{V}$ satisfying*

$$\begin{aligned} x &\approx p_0(x,y,z) \\ p_i(x,y,x) &\approx x \text{ for all } i \\ p_i(x,y,y) &\approx p_{i+1}(x,y,y) \text{ for even } i \\ p_i(x,x,y) &\approx p_{i+1}(x,x,y) \text{ for odd } i \\ p_n(x,y,y) &\approx q(x,y,y) \\ q(x,x,y) &\approx y \end{aligned}$$

3. Minimal algebras and tame quotients

This section is mainly a review of the very basic elements of tame congruence theory, based on [5]. The definition and statements are for quasiorders, though. In this section, $\mathbf{A}$ is always a finite algebra, and $\alpha < \beta$ are quasiorders of it.

Definition 3.1. A pair of elements (l_1, l_2) of a lattice is called a *quotient* of that lattice if $l_1 < l_2$, and a *prime quotient* if $l_1 \prec l_2$. If $l_1 \leq l_3 < l_4 \leq l_2$, then (l_3, l_4) is a *subquotient* of (l_1, l_2).

Definition 3.2. A set $U \subseteq A$ is (α,β)-*minimal* if there is a unary polynomial p such that $p(A) = U$ and $p(\beta) \nsubseteq \alpha$ (that is, there exists $(x,y) \in \beta$ with $(p(x), p(y)) \notin \alpha$), but there is no $q \in \mathrm{Pol}_1\,\mathbf{A}$ such that $q(A) \subsetneq U$ and $q(\beta) \nsubseteq \alpha$.

The set of all (α,β)-minimal sets of $\mathbf{A}$ is denoted by $M(\alpha,\beta)$.
$\mathbf{A}$ is an (α,β)-*minimal algebra* if A is an (α,β)-minimal set.
Finally, $\mathbf{A}$ is considered (γ,γ)-minimal for all $\gamma \in \operatorname{Quo}\mathbf{A}$.

The last part of the definition was only mentioned because technically, (γ,γ) is not a quotient of $\operatorname{Quo}\mathbf{A}$. It is completely in line with the rest of the definition otherwise.

Definition 3.3. For any set $U \subseteq A$, the algebra $\mathbf{A}|_U$ is an algebra with underlying set U, whose set of basic operations is the set of all polynomials of $\mathbf{A}$ to which U is closed.

For a binary relation δ on A, $\delta|_U$ denotes the binary relation $\delta \cap U^2$ on U. Sometimes, if it does not cause confusion, I write δ instead of $\delta|_U$.

Proposition 3.4. *For any $U \subseteq A$, $\mathbf{A}|_U$ is an algebra in which any term and any polynomial is a basic operation.*

If δ is compatible and reflexive in $\mathbf{A}$, then $\delta|_U$ is compatible on $\mathbf{A}|_U$. Thus $\delta \mapsto \delta|_U$ induces a mapping from $\operatorname{Quo}\mathbf{A}$ to $\operatorname{Quo}\mathbf{A}|_U$ (and from $\operatorname{Con}\mathbf{A}$ to $\operatorname{Con}\mathbf{A}|_U$), these mappings are meet homomorphisms.

If there is an idempotent unary polynomial e such that $e(A) = U$, then $\delta \mapsto \delta|_U$ is a surjective lattice homomorphism from $\operatorname{Quo}\mathbf{A}$ to $\operatorname{Quo}\mathbf{A}|_U$.

Proposition 3.5. *If U is an (α,β)-minimal set, then $\mathbf{A}|_U$ is an (α,β)-minimal algebra.*

The next lemma is immediate from the definition of minimality.

Lemma 3.6. *Suppose $\mathbf{A}$ is finite, $\alpha,\beta \in \operatorname{Quo}\mathbf{A}$ such that $\alpha < \beta$ and $\mathbf{A}$ is (α,β)-minimal. Then $\mathbf{A}$ is also*

- *(α^{-1},β^{-1})-minimal.*
- *$(\alpha\wedge\gamma,\beta\wedge\delta)$-minimal and $(\alpha\vee\gamma,\beta\vee\delta)$-minimal for any $\gamma,\delta \in \operatorname{Quo}\mathbf{A}$ such that $\mathbf{A}$ is also (γ,δ)-minimal.*
- *(γ,δ)-minimal whenever (γ,δ) is a subquotient of (α,β).*

□

Lemma 3.7. *Suppose that an algebra $\mathbf{A}$ is minimal with respect to one of its quasiorder quotients. Then $\mathbf{A}$ is also minimal with respect to either a congruence quotient or a quotient whose quasiorders have coinciding congruence parts.*

Proof. Choose $\beta \in \operatorname{Quo}\mathbf{A}$ so that there is a quasiorder α such that $\mathbf{A}$ is (α,β)-minimal, and β is minimal among such quasiorders. According to the previous lemma, for any $\gamma \in \operatorname{Quo}\mathbf{A}$ either $\gamma \geq \beta$ or $\alpha\wedge\gamma = \beta\wedge\gamma$.

If β is a congruence, take $\gamma = \alpha^{-1} < \beta$ to deduce $\alpha\wedge\alpha^{-1} = \beta\wedge\alpha^{-1} = (\beta\wedge\alpha)^{-1} = \alpha^{-1}$, whence α is a congruence. If β is not a congruence, choosing $\gamma = \beta^{-1}$ yields that the congruence part of β is in α. □

Definition 3.8. The pair (α,β) is called a *quasiorder tame quotient* (*congruence tame quotient*) if there is an (α,β)-minimal set U and an idempotent unary polynomial e such that $e(A) = U$, and $\alpha|_U < \delta|_U < \beta|_U$ for all $\alpha < \delta < \beta$ in $\operatorname{Quo}\mathbf{A}$ (in $\operatorname{Con}\mathbf{A}$).

The following is parts of Theorems 2.8. and 2.11. of [5] stated to quasiorders. The proofs there can be applied word-for-word, as they do not use symmetry.

Theorem 3.9. *If $\alpha \prec \beta$ in* Quo **A***, then (α,β) is tame.*

If (α,β) is tame, and U and V are (α,β)-minimal sets, then there is an idempotent unary polynomial e such that $e(A)=U$ and $e(\beta)\nsubseteq\alpha$. □

The following is a not-so basic element of tame congruence theory (see Theorem 8.5, Lemma 4.17 and Lemma 4.20 of [5]).

Lemma 3.10. *Let* **A** *be a finite algebra in a congruence modular variety. If* **A** *is minimal to a congruence prime quotient, then it either is a two-element algebra, or has a Mal'tsev polynomial.*

4. Quasiorder lattices in congruence modular varieties

Lemma 4.1. *Suppose that* **A** *is a finite algebra in a congruence modular variety that is (α,β)-minimal for quasiorders $\alpha<\beta$, where $\alpha^*=\beta^*$. Then β^* has exactly two blocks.*

Proof. As β is not a congruence, there are elements $a,b\in A$ such that $a/\beta^*\prec_{\beta/\beta^*} b/\beta^*$ and $(a,b)\notin\alpha$. As **A** is in a congruence modular variety, it admits Gumm terms $p_1,\dots,p_n,q$.

For each $1\le i\le n$,

$$a=p_i(a,a,a)\xrightarrow{\beta}p_i(a,a,b)\xrightarrow{\beta}p_i(a,b,b)\xrightarrow{\beta}p_i(b,b,b)=b,$$

so both $p_i(a,a,b)$ and $p_i(a,b,b)$ is in the β^*-block of either a or b. Notice that there must be a j so that $p_j(a,a,b)$ and $p_j(a,b,b)$ are in different β^*-blocks, otherwise by the Gumm identities $p_n(a,b,b)$ would be in the β^*-block of a, which contradicts $b=q(a,a,b)\xrightarrow{\beta}q(a,b,b)=p_n(a,b,b)$.

The unary polynomial $p_j(a,x,b)$ thus maps (a,b) to a $\beta\backslash\alpha$-edge (as $\alpha^*=\beta^*$, any edge with source in the β^*-block of a and target in the β^*-block of b must not be in α). By the (α,β)-minimality of **A**, $p_j(a,x,b)$ must be a bijective polynomial, but as

$$a=p_j(a,x,a)\xrightarrow{\beta}p_j(a,x,b)\xrightarrow{\beta}p_j(b,x,b)=b,$$

this polynomial maps A to the union of the β^*-blocks of a and b. Thus β^* only has two blocks. □

Lemma 4.2. *Let* **A** *be a finite algebra in a congruence modular variety. If* **A** *is minimal to a quasiorder quotient, then it either is a two-element algebra, or has a Mal'tsev polynomial.*

Proof. By Lemma 3.7, **A** is minimal to either a congruence quotient or a quasiorder $+$-quotient (α,β). In the second case, by Lemma 4.1, β^* has two blocks. This is only possible if $\alpha=\beta^*$, but then by Lemma 3.6, **A** is also $(\alpha,1_{\mathbf{A}})$-minimal.

Hence, **A** is necessarily minimal to a congruence quotient, and obviously, it must then be minimal to a congruence prime quotient. By Lemma 3.10, the proof is done. □

There is one more ingredient I need: when substituting an algebra into the one induced by a minimal set (with respect to some quasiorder prime quotient), one does not leave the class of algebras generating a congruence modular variety.

Proposition 4.3. *If* **A** *admits Gumm-terms, $\alpha\prec\beta$ in* Quo **A***, and U is an (α,β)-minimal algebra, then $\mathbf{A}|_U$ also admits Gumm-terms.*

Proof. (α, β) is tame by Theorem 3.9, so by Proposition 3.4, there is an idempotent unary polynomial e of $\mathbf{A}$ such that $e(A) = U$. For any k-ary polynomial t of $\mathbf{A}$ the k-ary polynomial $e(t)$ is defined by

$$e(t)(x_1, \dots, x_k) = e(t(x_1, \dots, x_k)),$$

this is a *term* on $\mathbf{A}|_U$. Therefore, if $p_0, \dots, p_n, q$ are Gumm-terms on $\mathbf{A}$, then $e(p_0), \dots, e(p_n), e(q)$ are Gumm-terms on $\mathbf{A}|_U$. □

Theorem 4.4. *Let* $\mathbf{A}$ *an algebra in a locally finite congruence modular variety, and denote with* $\overline{\delta}$ *the transitive closure of a compatible reflexive relation* δ *on* A*. The equality* $\overline{\rho \cap \sigma} = \overline{\rho} \cap \overline{\sigma}$ *is satisfied for arbitrary reflexive compatible relations* ρ, σ *of* $\mathbf{A}$*. Thus taking transitive closures induces a homomorphism from the lattice of compatible reflexive relations of* $\mathbf{A}$ *to* $\operatorname{Quo} \mathbf{A}$*.*

Proof. Suppose $\overline{\rho \cap \sigma} < \overline{\rho} \cap \overline{\sigma}$. It can be assumed that $\mathbf{A}$ is finite, as if (a, b) is an element of the right side and not of the left, there are elements $c_1, \dots, c_k, d_1, \dots, d_l \in A$ such that $a \xrightarrow{\rho} c_1 \xrightarrow{\rho} \dots \xrightarrow{\rho} c_k \xrightarrow{\rho} b$ and $a \xrightarrow{\sigma} d_1 \xrightarrow{\sigma} \dots \xrightarrow{\sigma} d_l \xrightarrow{\sigma} b$, and the elements $a, b, c_1, \dots, c_k, d_1, \dots, d_l$ generate a finite counterexample.

Take a $\nu \in \operatorname{Quo} \mathbf{A}$ so that $\overline{\rho \cap \sigma} \prec \nu \leq \overline{\rho} \cap \overline{\sigma}$. It can be assumed that $\mathbf{A}$ is a $(\overline{\rho \cap \sigma}, \nu)$-minimal algebra, because otherwise, its restriction to a minimal set will yield a counterexample of smaller cardinality.

By Lemma 4.2, $\mathbf{A}$ is either a two-element algebra, or has a Mal'tsev polynomial. The first case is impossible: it is very easy to see that this theorem does not have a two-element counterexample. In the second case, all the reflexive compatible relations of $\mathbf{A}$ are tolerances: By Theorem 2 of [1] (what this theorem generalizes), this is a contradiction. □

Theorem 4.5. *Suppose that* $\mathbf{A}$ *is a finite algebra in a congruence modular variety. Then* $\operatorname{Con} \mathbf{A}$ *and* $\operatorname{Quo} \mathbf{A}$ *satisfy the same lattice identities.*

Proof. Obviously, any identity satisfied by $\operatorname{Quo} \mathbf{A}$ is also satisfied by $\operatorname{Con} \mathbf{A}$. Suppose that the converse is not true, that there is a lattice identity $p \approx q$ that holds in $\operatorname{Con} \mathbf{A}$, and does not hold in $\operatorname{Quo} \mathbf{A}$. I will assume two things. Firstly, that $p \leq q$ is an identity that holds in all lattices (and so $p \approx q$ is equivalent to $p \not< q$). Secondly, that $\mathbf{A}$ is a minimal counterexample, in the sense that for every $\mathbf{B}$ with smaller cardinality, if $\mathbf{B}$ lies in a congruence modular variety, and $\operatorname{Con} \mathbf{B}$ satisfies $p \approx q$, then $\operatorname{Quo} \mathbf{B}$ also satisfies $p \approx q$.

The fact that $p \approx q$ is not satisfied by $\operatorname{Quo} \mathbf{A}$ means that there are quasiorders $\alpha_1, \dots, \alpha_n, \mu, \nu$ of $\mathbf{A}$ such that

$$p(\alpha_1, \dots, \alpha_n) \leq \mu \prec \nu \leq q(\alpha_1, \dots, \alpha_n)$$

holds in $\operatorname{Quo} \mathbf{A}$ (p and q are assumed to be n-ary). For a (μ, ν)-minimal set U, the algebra $\mathbf{A}|_U$ is in a congruence modular variety by Proposition 4.3, $\operatorname{Quo} \mathbf{A}|_U$ does not satisfy $p \approx q$, but $\operatorname{Con} \mathbf{A}|_U$ does (because it is a homomorphic image of $\operatorname{Con} \mathbf{A}$ by Proposition 3.4 and Theorem 3.9).

Therefore, by the minimality assumption, $\mathbf{A}$ must be (μ, ν)-minimal. By Lemma 4.2, it is either a two-element algebra or has a Mal'tsev polynomial. Both are impossible. In the first case the congruence lattice of the algebra is isomorphic to the two-element lattice, and the quasiorder lattice is isomorphic either to the same, or to its direct square, so they satisfy the same identities. In the second case, $\operatorname{Quo} \mathbf{A} = \operatorname{Con} \mathbf{A}$ by Proposition 2.1. □

Corollary 4.6. *Suppose that $\mathcal{P}$ is a lattice identity so that each variety whose congruence lattices satisfy $\mathcal{P}$ is congruence modular. Then if all congruence lattices of a locally finite variety satisfy $\mathcal{P}$, then so do all the quasiorder lattices of the variety.*

I note that the condition here for $\mathcal{P}$ is weaker then the condition that it should be a stronger lattice identity than modularity. For example, the so-called *Arguesian* identity is a *weaker* lattice identity than modularity, but a variety is congruence Arguesian precisely if it is congruence modular (see [6]).

Problem 4.7. For which lattice identities is it true that if the congruence lattices of a locally finite variety satisfy it, then so do the quasiorder lattices of the variety? Does the answer change without assuming locally finiteness? In particular, is it true that for any lattice identity stronger than modularity, if the congruence lattices of the variety satisfy it then so do the quasiorder lattices?

Problem 4.8. Is Corollary 4.6 true for quasi-identities?

Problem 4.9. Is there a general way of obtaining $\operatorname{Quo}\mathbf{A}$ from $\operatorname{Con}\mathbf{A}$ for a finite $\mathbf{A}$ in a congruence modular variety using the H, S, P operators? (According to 4.5, they are in the same lattice variety.)

I note that the answer to the last problem is given in [2] for lattices: the quasiorder lattice of a lattice is isomorphic the direct square of the congreuence lattice.

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