n-PERMUTABILITY IS NOT JOIN-PRIME FOR $n \geq 5$

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ABSTRACT. Let $\mathscr{V}_P$ be the variety generated by an order primal algebra of finite signature associated with a finite bounded poset P that admits a near-unanimity operation. Let Λ be a finite set of linear identities that does not interpret in $\mathscr{V}_P$. Let $\mathscr{V}_\Lambda$ be the variety defined by Λ. We prove that $\mathscr{V}_P \vee \mathscr{V}_\Lambda$ is $\boldsymbol{n}$-permutable for some $\boldsymbol{n}$. This implies that there is an $\boldsymbol{n}$ such that $\boldsymbol{n}$-permutability is not join-prime in the lattice of interpretability types of varieties. In fact, it follows that $\boldsymbol{n}$-permutability where $\boldsymbol{n}$ runs through the integers greater than 1 is not prime in the lattice of interpretability types of varieties.

We strengthen this result by making P and Λ more special. We let P be the 6-element bounded poset that is not a lattice and $\mathscr{V}_m$ the variety defined by the set of majority identities for a ternary operational symbol m. We prove in this case that $\mathscr{V}_P \vee \mathscr{V}_m$ is 5-permutable. This implies that $\boldsymbol{n}$-permutability is not join-prime in the lattice of interpretability types of varieties whenever $\boldsymbol{n} \geq 5$. We also provide an example demonstrating that $\mathscr{V}_P \vee \mathscr{V}_m$ is not 4-permutable.

1. INTRODUCTION

Let Γ be a set of identities over a certain signature of a variety. We say that Γ *interprets in a variety* $\mathscr{K}$ if by replacing the operation symbols in Γ by term expressions of $\mathscr{K}$—same symbols by same terms with arities kept—the so obtained set of identities holds in $\mathscr{K}$. A *variety* $\mathscr{K}_1$ *interprets in a variety* $\mathscr{K}_2$ if there is a set of identities Γ that defines $\mathscr{K}_1$ and interprets in $\mathscr{K}_2$. Roughly speaking, a variety $\mathscr{K}_1$ interprets in a variety $\mathscr{K}_2$ if $\mathscr{K}_2$ has a richer algebraic structure than $\mathscr{K}_1$. Nevertheless, we have to be cautious with this rough approach, since, for example, the variety of sets with no basic operations and the variety of semigroups are *equi-interpretable*, meaning that they interpret in each other.

As easily seen, interpretability is a quasiorder and equi-interpretability is an equivalence on the class of varieties. The blocks of equi-interpretability are called the *interpretability types*. In [3] Garcia and Taylor introduced the *lattice of interpretability types of varieties* that is obtained by taking the quotient of the class of varieties quasiordered by interpretability and the equi-interpretabiliy relation. The join in this lattice is described as follows. Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be two varieties of disjoint signatures. Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be defined by the sets Σ_1 and Σ_2 of identities, respectively. *Their join* $\mathscr{K}_1 \vee \mathscr{K}_2$ is the variety defined by $\Sigma_1 \cup \Sigma_2$. The so defined join is compatible with the interpretability relation of varieties, and naturally yields the definition of the join operation in the lattice of interpretability types of varieties.

Let $\boldsymbol{n} \geq 2$ be an integer. An *algebra* $\mathbf{A}$ *is congruence* $\boldsymbol{n}$*-permutable*, if for any two congruences α and β of $\mathbf{A}$, $\alpha\beta\cdots = \beta\alpha\ldots$ where each side of the equality

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consists of $\boldsymbol{n}$ alternating factors of α and β. An *algebra* $\mathbf{A}$ *is congruence distributive (congruence modular)*, if for any three congruences α, β and γ of $\mathbf{A}$ (with $\alpha \leq \gamma$)

$$(\alpha \vee \beta) \wedge \gamma = (\alpha \wedge \gamma) \vee (\beta \wedge \gamma).$$

A *variety is* $\boldsymbol{n}$*-permutable (distributive, modular)* if all of its members are congruence $\boldsymbol{n}$-permutable (distributive, modular). An *n-ary operation f is idempotent* if it satisfies the identity $f(x,x,\ldots,x) = x$. A *variety is idempotent* if all of its members have idempotent basic operations.

In [3] Garcia and Taylor formulated the conjectures that 2-permutability is join-prime and the interpretability types of the modular varieties form a prime filter in the lattice of interpretability types of varieties. In [15] Tschantz announced a proof of the conjecture on 2-permutability. However, his proof has remained unpublished.

For restricted versions of the conjectures some positive results have been attained. Following the pioneering work of Hobby and McKenzie on locally finite varieties in [8], Kearnes and Kiss stepped beyond locally finiteness and gave a classification of varieties in [9]. Their results imply that certain idempotent Mal'cev classes identified in their work form a prime filter in the lattice of interpretability types of idempotent varieties. In [16] Valeriote and Willard, by supplementing the characterization of a Mal'cev class in [9], proved that the interpretability types of the idempotent $\boldsymbol{n}$-permutable varieties for $\boldsymbol{n} \geq 2$ form also a prime filter in the idempotent case. In [14] Opršal obtained a similar result for idempotent modular varieties. In [10] Kearnes and Szendrei proved that for any $\boldsymbol{n}$ having an $\boldsymbol{n}$-cube term is a join-prime property in the idempotent case.

In the present note we give some negative results related to the conjecture on permutability in the general case (where idempotency is not assumed). We shall prove that the filter of the interpretability types of the $\boldsymbol{n}$-permutable varieties where $\boldsymbol{n}$ runs through the integers greater than 1 is not prime in the lattice of interpretability types of varieties. We shall also prove that for any $\boldsymbol{n} \geq 5$, $\boldsymbol{n}$-permutability is not join-prime in the lattice of interpretability types of varieties.

2. $\boldsymbol{n}$-PERMUTABILITY FOR SOME $\boldsymbol{n}$

An n-ary operation f, $n \geq 3$, is a *near-unanimity operation* if it satisfies the identities

$$f(y,x,\ldots,x) = f(x,y,\ldots,x) = \cdots = f(x,x,\ldots,y) = x.$$

A ternary near-unanimity operation is called a *majority operation*. Clearly, the near-unanimity operations are idempotent. It is well known that on a finite set any clone that contains an n-ary near-unanimity operation is finitely generated. It is also known that any algebra that has a near-unanimity term operation is congruence distributive.

Let P denote a finite poset. Let $\mathbf{P}$ be an algebra whose underlying set equals that of P and whose basic operations form a generating set of the clone of P. We call such an algebra an *order primal* algebra with respect to P. Let $\mathcal{V}_P$ denote the variety generated by $\mathbf{P}$. If P admits a near-unanimity operation, then the clone of monotone operations of P is finitely generated, and so the algebra $\mathbf{P}$ can be chosen to be of finite signature. We always make this choice throughout the present paper for any finite poset P when P admits a near-unanimity operation. Then by Baker's finite basis theorem in [1] for finite algebras of finite signature in a congruence distributive variety, there exists a finite set Σ of identities that serves as a finite basis for $\mathcal{V}_P$.

An identity is called a *linear identity* if on both sides it has at most one operation symbol. Let P be a finite bounded poset that admits a near-unanimity operation and Λ a finite set of linear identities that does not interpret in $\mathcal{V}_P$. We assume that Λ is given in a signature disjoint from that of $\mathcal{V}_P$. Let $\mathcal{V} = \mathcal{V}_P \vee \mathcal{V}_\Lambda$, so $\mathcal{V}$ is the variety that is defined by the identities $\Sigma \cup \Lambda$. A *term reduct of an algebra* $\mathbf{A}$ is an algebra whose underlying set coincides with that of $\mathbf{A}$ and whose basic operations are term operations of $\mathbf{A}$. We note that by Jónsson's theorem for congruence distributive varieties in [2], every algebra in $\mathcal{V}$ has a term reduct that is isomorphic to a subdirect power of $\mathbf{P}$ where $\mathbf{P}$ is an order primal algebra for P. Later in the proof of our main result we use this observation.

Our aim is to prove in this section that $\mathcal{V}$ is $\boldsymbol{n}$-permutable for some $\boldsymbol{n}$. It follows immediately that there exists an $\boldsymbol{n}$ such that $\boldsymbol{n}$-permutability is not join-prime in the lattice of interpretability types of varieties. For the proof of the main result of this section we require two simple propositions.

Proposition 2.1. *Let Q be finite bounded poset, and $\mathbf{Q}$ an order-primal algebra determined by Q. The compatible quasiorders of $\mathbf{Q}$ are the equality, the full relation, $\leq$ and $\geq$.*

Proof. It is enough to prove that the quasiorders generated by a single pair are of the above form. Let p and q be two different elements of Q. When $p \leq q$, then (p,q) clearly generates $\leq$. Similarly, when $q \leq p$, then (p,q) generates $\geq$. Finally, observe that any pair (p,q) where p and q are incomparable in Q can be mapped into any pair by a monotone map, since Q is bounded. Thus the quasiorder generated by such a pair (p,q) is the full relation. □

Proposition 2.2. *Let P be a finite bounded poset that admits a near-unanimity operation. Let $\mathbf{D} \leq \mathbf{P}^n$ and δ a compatible quasiorder of $\mathbf{D}$. Then δ is a* product quasiorder*, i.e., there are compatible quasiorders $\delta_1, \dots, \delta_n$ of $\mathbf{P}$ such that for all $a = (a_1, \dots, a_n)$ and $a' = (a'_1, \dots, a'_n)$ in D*

$$(a, a') \in \delta \Leftrightarrow \forall i\colon (a_i, a'_i) \in \delta_i.$$

Proof. For any $1 \leq i \leq n$, let η_i denote the kernel of the projection from $\mathbf{D}$ to the i-th coordinate. By Theorem 2.6 in [5], the quasiorder lattice of $\mathbf{D}$ is a distributive lattice, so

$$\delta = \delta \vee 0_{\mathbf{B}} = \delta \vee \left(\bigwedge \eta_i\right) = \bigwedge (\delta \vee \eta_i).$$

Since $\mathbf{P}$ has no proper subalgebras and is simple, $\mathbf{D}/\eta_i \cong \mathbf{P}$. Hence the quotient of $\delta \vee \eta_i$ by η_i corresponds to a compatible quasiorder δ_i of $\mathbf{P}$ by the correspondence theorem of quasiorders. This gives that for every i

$$(a, a') \in \delta \vee \eta_i \Leftrightarrow (a_i, a'_i) \in \delta_i.$$

□

An n-ary *compatible relation* of an algebra $\mathbf{A}$ is a subuniverse of $\mathbf{A}^n$. Let D be an n-ary compatible relation of $\mathbf{P}$. A *representation of* D is a pair (R,S) where R is a finite quasiordered set, S is an n-element subset of R and there exists an enumeration $s_1, \dots, s_n$ of the elements of S such that

$$D = \{\, (f(s_1), \dots, f(s_n)) \mid f\colon R \to P \text{ is a monotone map} \,\}.$$

Notice that if (R,S) and (R',S') are two representations of D (even with possibly different base sets) and P is not an antichain, then the restrictions of R to S and R' to $S' = \{s'_1,\dots,s'_n\}$ are isomorphic via the map $s_i \mapsto s'_i$. Indeed, for any $s_i, s_j \in S$, $s_i \leq s_j$ if and only if for all monotone $f\colon R \to P$, $f(s_i) \leq f(s_j)$, and the latter condition only depends on the projections of D onto its i and j coordinates. We note that every finitary compatible relation of $\mathbf{P}$ has a representation obtained in an obvious way from a primitive positive formula defining the relation in the language $\{\leq_P\}$, see [4].

Let C and D be two n-ary compatible relations of $\mathbf{P}$ such that $C \subseteq D$. Then every representation (R,S) of D extends to a representation (T,S) of C by adding suitable vertices and edges to (R,S). Indeed, if (R,S) is a representation of D and (R',S) is an arbitrary representation of C such that the only elements shared by R and R' are the elements of S, then T is obtained from $R \cup R'$ by taking the transitive closure of the relation of $R \cup R'$.

Let Q and R be two quasiordered sets. A monotone map $g\colon Q \to R$ is called a *retraction* if there is a monotone map $h\colon R \to Q$ such that $gh = \mathrm{id}_R$. Then the map h is called a *coretraction*. If there is retraction from Q to R, then R is called a *retract* of Q. Now we present the main result of the section.

Theorem 2.3. *Let P be a finite bounded poset that admits a near-unanimity operation and $\mathbf{P}$ an associated order primal algebra of finite signature. Let $\mathscr{V}_P$ denote the variety generated by $\mathbf{P}$. Let Σ be a finite basis for $\mathscr{V}_P$ and Λ a finite set of linear identities in a disjoint signature. If Λ does not interpret in the variety $\mathscr{V}_P$, then the variety $\mathscr{V}$ defined by $\Sigma \cup \Lambda$ is $\boldsymbol{n}$-permutable for some $\boldsymbol{n}$.*

Proof. In order to prove the statement we invoke a theorem of Hagemann in [6], an implicit proof is given by Lakser in [12]: a variety is $\boldsymbol{n}$-permutable for some $\boldsymbol{n}$ if and only if every compatible quasiorder of any algebra of the variety is a congruence. So it suffices to prove that every compatible quasiorder of any algebra in $\mathscr{V}$ is a congruence. Let us suppose to the contrary that there is an algebra $\mathbf{B}$ in $\mathscr{V}$ that has a compatible quasiorder δ that is not a congruence. As we mentioned earlier, by Jónsson's theorem, without loss of generality we may assume that B is a subuniverse of $\mathbf{P}^I$ for some set I.

We pick an edge (y_0,z_0) in the quasiordered set (B,δ) such that $(z_0,y_0) \notin \delta$. We define two subalgebras of $\mathbf{P}^I$ whose underlying sets are contained in B. Let $\mathbf{C}_0$ be the subalgebra generated by $\{y_0,z_0\}$ in $\mathbf{P}^I$, and let $\mathbf{D}_0$ be the subalgebra generated by

$$C_0 \cup \{\, f(u_1,\dots,u_k) \mid f \text{ is an operation symbol of arity } k \text{ in } \Lambda,\ u_1,\dots,u_k \in C_0 \,\}$$

also in $\mathbf{P}^I$. Clearly, y_0 and z_0 are in $\mathbf{C}_0$, and $C_0 \subseteq D_0 \subseteq B$. Since $\mathbf{C}_0$ and $\mathbf{D}_0$ are finitely generated algebras in a locally finite variety $\mathscr{V}_P$, both $\mathbf{C}_0$ and $\mathbf{D}_0$ are finite subalgebras of $\mathbf{P}^I$. Hence there is a finite subset of I such that the projection of $\mathbf{D}_0$ to this subset is a bijective map. Let y, z, $\mathbf{C}$ and $\mathbf{D}$ be the images of y_0, z_0, $\mathbf{C}_0$ and $\mathbf{D}_0$ under this projection, respectively. The restrictions of δ to $\mathbf{C}$ and $\mathbf{D}$ project down to the quasiorders $\delta_{\mathbf{C}}$ and $\delta_{\mathbf{D}}$, respectively. So $\mathbf{C}$ and $\mathbf{D}$ are subalgebras of $\mathbf{P}^m$ for some finite m, and $\delta_{\mathbf{C}}$ and $\delta_{\mathbf{D}}$ are compatible quasiorders of $\mathbf{C}$ and $\mathbf{D}$, respectively. Moreover, $(y,z) \in \delta_{\mathbf{C}} \subseteq \delta_{\mathbf{D}}$ and $(z,y) \notin \delta_{\mathbf{D}}$.

Let (T,S) and (R,S) be representations of the m-ary compatible relations C and D, respectively. Without loss of generality—by the note preceding the present theorem—we assume that $R \subseteq T$ where containment means that (T,S) is obtained from (R,S) by adding suitable vertices and edges. The quasiorder $\delta_{\mathbf{D}}$ is not a congruence of $\mathbf{D}$, and by Proposition 2.2, $\delta_{\mathbf{D}}$ is a product quasiorder. So there exist quasiorders δ_i, $i \in S$, as in Proposition 2.2. Let $r \in S$ such that $\delta_r = \leq_P$ or $\delta_r = \geq_P$ and, in both cases, the r-th coordinates of y and z are different. There exists such an r, since $(y,z) \in \delta_{\mathbf{D}}$ and $(z,y) \notin \delta_{\mathbf{D}}$. For $\delta_{\mathbf{C}}$ there also exist quasiorders δ_i', $i \in S$, witnessing the claim in Proposition 2.2. Notice that $\delta_i' \subseteq \delta_i$, $i \in S$, and hence $\delta_r' = \delta_r$. Without loss of generality we may assume $\delta_r = \leq_P$.

Let g be the r-th projection from D to P. We prove that g is a retraction from the quasiordered set $(D, \delta_{\mathbf{D}})$ onto the poset P. Certainly, g is monotone. We define a map h from P to C: for any $p \in P$ let $h(p) = (h_1(p), \ldots, h_m(p))$, where for any $i \in T$

$$h_i(p) := \begin{cases} p, & \text{if } i \leq r \leq i \text{ in } T, \\ 1, & \text{if } i \not\leq r \leq i \text{ in } T, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that the sequences $(h_i(p))_{i \in T}$, $p \in P$, are monotone maps from T to P. Therefore, for all $p \in P$, $h(p) \in C$. Moreover, the sequences $(h_i(p))_{i \in T}$, $p \in P$, differ only on the r-block of T where they are defined to equal the constant p, respectively. So by applying Proposition 2.2 and taking into account that $\delta_r = \leq_P$, we get that $h(p) \xrightarrow{\delta} h(q)$ for all $p \leq q$ in P. Hence h is monotone. Obviously, $gh = \mathrm{id}_P$ by the definitions of g and h. Thus, g is indeed a retraction. Moreover, h is a corresponding coretraction with the property that its image is contained in C.

By the facts proved in the preceding two paragraphs, since $(D_0, \delta|_{D_0})$ is isomorphic to $(D, \delta_{\mathbf{D}})$ under an isomorphism that maps $(C_0, \delta|_{C_0})$ to $(C, \delta_{\mathbf{C}})$, there exist a retraction g_0 from $(D_0, \delta|_{D_0})$ onto P and a corresponding coretraction h_0 from P into $(D_0, \delta|_{D_0})$ such that $h_0(P) \subseteq C_0$.

It is well known that linear identities are preserved under taking retract. Basically, the proof of this fact will work in our case as well to get that Λ interprets in $\mathscr{V}_P$. Here is the proof adapted to our situation. For any operation symbol f in Λ we define a term operation t_f on $\mathbf{P}$ by

$$t_f(x_1, \ldots, x_k) := g_0(f(h_0(x_1), \ldots, h_0(x_k)))$$

provided f is of arity k. The operation t_f is well-defined, as $h_0(P) \subseteq C_0$ and by the definition of $\mathbf{D}_0$, f maps k-tuples over C_0 into D_0. Moreover, t_f is monotone on P, since g_0, h_0 and f are monotone. Now the t_f satisfy the identities of Λ on P, since the f satisfy them on the tuples over C_0. Thus Λ interprets in $\mathscr{V}_P$, which contradicts our assumption on Λ. □

We conjecture that the statement of the preceding theorem extends to every finite bounded poset P where $\mathbf{P}$ generates a join semi-distributive variety.

For the rest of the paper we let P denote the poset with underlying set $\{0, a, b, c, d, 1\}$ and covering relation

$$0 \prec a, b \prec c, d \prec 1,$$

see the first poset in Figure 1. It is well known that P admits a 5-ary but no 4-ary near-unanimity operation, see [17]. Let Λ be a finite basis for the variety $\mathscr{V}_Q$ where Q is a two-element chain. Clearly, Λ does not interpret $\mathscr{V}_P$, since Q admits a majority operation and P does not. Then by the preceding theorem $\mathscr{V}_P \vee \mathscr{V}_Q$ is $\boldsymbol{n}$-permutable for some $\boldsymbol{n}$. On the other hand, by Hagemann's above mentioned

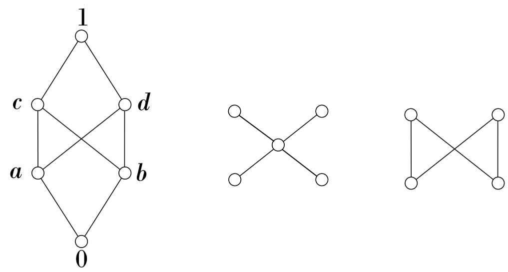


FIGURE 1. Posets P, X and the 4-crown

result in [6], both of $\mathscr{V}_P$ and $\mathscr{V}_Q$ are not $\boldsymbol{n}$-permutable for any $\boldsymbol{n}$. Thus we have the following corollaries.

Corollary 2.4. *The filter of the interpretability types of the* $\boldsymbol{n}$*-permutable varieties where* $\boldsymbol{n}$ *runs through the integers greater than 1 is not prime in the lattice of interpretability types of varieties*

Corollary 2.5. *There is an* $\boldsymbol{n}$ *such that congruence* $\boldsymbol{n}$*-permutability is not join-prime in the lattice of interpretability types of varieties.*

3. 5-PERMUTABILITY

Recall that P denotes the six-element poset in Figure 1. Let m be a ternary operation symbol, and let $\mathscr{V}_m$ be the variety defined by the set Λ of majority identities $m(y,x,x) = m(x,y,x) = m(x,x,y) = x$ for m. We saw that Λ does not interpret in $\mathscr{V}_P$. We let $\mathscr{V} = \mathscr{V}_P \vee \mathscr{V}_m$. In this section we shall prove that $\mathscr{V}$ is 5-permutable, and hence for any $\boldsymbol{n} \geq 5$ congruence $\boldsymbol{n}$-permutability is not join-prime in the lattice of interpretability types of varieties.

In [7] Hagemann and Mitschke proved that for a given $\boldsymbol{n}$, $\boldsymbol{n}$-permutability of a variety is a strong Mal'cev condition. Moreover, they gave the following characterization of $\boldsymbol{n}$-permutability of a variety.

Theorem 3.1 (Hagemann, Mitschke (1973))**.** *Let* $\mathscr{K}$ *be a variety and* $\boldsymbol{n} \geq 2$ *an integer. Then the following are equivalent.*

(1) $\mathscr{K}$ *is* $\boldsymbol{n}$*-permutable.*
(2) *Any edge of a reflexive compatible binary relation* ρ *of any algebra* $\mathbf{A} \in \mathscr{K}$ *is in a directed* $\boldsymbol{n}$*-cycle of the digraph* (A,ρ).

Let $\mathbf{B}$ be an algebra in variety $\mathscr{V}$, and ρ a binary reflexive compatible relation of $\mathbf{B}$. In this section we are going to prove that every edge from ρ is in a directed cycle of length 5 in the digraph (B,ρ). If this is done, the above result of Hagemann and Mitschke yields that $\mathscr{V}$ is 5-permutable. We already saw that neither variety $\mathscr{V}_P$ nor the variety generated by an order primal algebra associated with the two element chain are congruence $\boldsymbol{n}$-permutable for any $\boldsymbol{n}$. On the other hand the join of these two varieties is 5-permutable. Thus $\boldsymbol{n}$-permutability is not join-prime in the lattice of interpretability types of varieties for any $\boldsymbol{n} \geq 5$.

A *colored digraph* (R,f) is a digraph R with a partial map f from R to P. The map f is called the *coloring of* (R,f), and the elements in the domain of f are called the *colored elements* of (R,f). If f extends to a fully defined edge preserving map from R to P, then (R,f) and f are called extendible. Most of the time we deal with colored quasiordered sets. A finite colored quasiordered set is called an *obstruction*, if it is non-extendible, but the restriction of its coloring to any quasiordered set properly contained in it is extendible. *Containment* here means subdigraph containment opposed to spanned subdigraph containment.

In the next lemma we characterize the obstructions among the finite colored quasiordered sets. The importance of this lemma is given by the fact that a finite colored quasiordered set is extendible if and only if it does not contain any obstruction. We introduce some types of obstructions before stating the lemma and launching into its proof. The simplest obstructions are the *twisted edges*, that is, 2-element chains with a fully defined non-monotone coloring. Let X be the 5-element poset that looks like an X, see the second item in Figure 1. If we color the minimal elements of X by a and b and its maximal elements by c and d, we obtain an obstruction called the *tight* X.

Lemma 3.2. *In the class of finite colored quasiordered sets the obstructions are the twisted edges and the tight* X.

Proof. Let (Q,f) be a finite colored quasiordered set. We assume that (Q,f) contains none of the twisted edges and a tight X. We then prove that f is extendible. Let α be the largest equivalence contained in the quasiorder of Q, and Q' the quotient poset Q/α. On each block of α, f must take on at most one value since there are no twisted edges contained in (Q,f). Hence f induces a partial map f' from the poset Q' to P. Observe that, by the assumption, the colored poset (Q',f') contains none of the twisted edges and a tight X. The statement of the present lemma was verified for the case when Q is a poset in [17], cf. the remark on fences on page 89 and Theorem 3.3. Hence f' extends to a fully defined monotone map from Q' to P. By composing a full extension of f' with the natural map from Q to Q' we obtain a required extension of f. $\square$

Let $\mathbf{D}$ be a subalgebra of $\mathbf{P}^n$ and ρ a reflexive compatible binary relation on $\mathbf{D}$. Then, we conceive ρ as a compatible $2n$-ary relation of $\mathbf{P}$, and hence it has a representation of the form $(R, S\cup S')$, where R is a quasiordered set,

$$S=\{s_1,\dots,s_n\},\ S'=\{s'_1,\dots,s'_n\}$$

are disjoint n-element subsets of R and

$$\rho=\{((f(s_1),\dots,f(s_n)),(f(s'_1),\dots,f(s'_n)))\mid f\colon R\to P \text{ is monotone}\}.$$

Now, by reflexivity, (R,S) and (R,S') are two representations of D, and the restrictions of R to the sets S and S' are isomorphic quasiordered sets via the map $s_i\mapsto s'_i$. Let $\mathbf{C}$ be a subalgebra of $\mathbf{D}$. Then any representation $(R,S\cup S')$ of ρ extends to a representation $(R,S\cup S')$ of $\rho|_C$ by adding suitable vertices and edges to $(R,S\cup S')$.

In the proof of our main result we require the following lemma.

Lemma 3.3. *Let* $(R,S\cup S')$ *be a representation of a binary reflexive relation* ρ *on* $\mathbf{D}$ *where* $\mathbf{D}$ *is a subalgebra of* $\mathbf{P}^n$ *where*

$$S=\{s_1,\dots,s_n\} \text{ and } S'=\{s'_1,\dots,s'_n\}.$$

(1) *If* $i,j\in S$ *and* $i\leq j'$ *in* R*, then* $i\leq j$ *and* $i'\leq j'$ *in* R.
(2) *Let* $i,j\leq k,l$ *in* S *and* $i^*\in\{i,i'\},\ j^*\in\{j,j'\},\ k^*\in\{k,k'\},\ l^*\in\{l,l'\}$*. If there is an* $r^*\in R$ *such that* $i^*,j^*\leq r^*\leq k^*,l^*$*, then there exist* $r,r^+\in R$ *such that* $i,j\leq r\leq k,l$ *and* $i',j'\leq r^+\leq k',l'$.

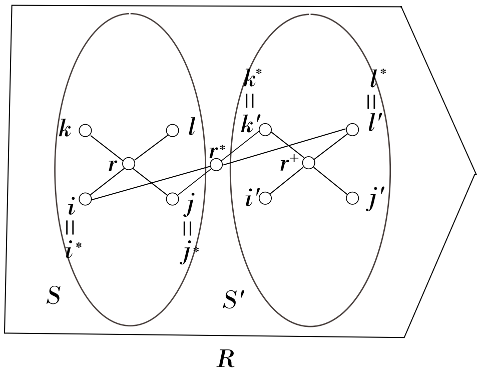


FIGURE 2. A figure presenting a particular case in the second statement of Lemma 3.3. It may happen that the vertices r and r^+ are outside of S and S', respectively.

Proof. Notice that if $i \not\leq j$ in R, then there is a monotone map $f\colon R \to P$ such that $f(i) = 1$ and $f(j) = 0$. Hence, for the first part of claim (1), it suffices to prove that for any monotone map $f\colon R \to P$, $f(i) \leq f(j)$. So let $f\colon R \to P$ be an arbitrary monotone map. Then $f|_S \in \mathbf{D}$. Since ρ is reflexive, if we color the vertices of both S and S' by $f|_S$ in R, we obtain an extendible colored quasiordered set. Suppose that $g\colon R \to P$ is an extension of the coloring of this colored quasiordered set. Then $f(i) = g(i) \leq g(j') = g(j) = f(j)$. Thus $i \leq j$. We similarly obtain that $i' \leq j'$.

For the first part of claim (2), we shall prove that there is an r such that $i, j \leq r \leq k, l$. If the sub quasiordered set spanned by the elements i, j, k, l is not a 4-crown—see the third item in Figure 1—we are clearly done. So we may assume that i, j, k, l spans a 4-crown in R. Suppose that there is no r such that $i, j \leq r \leq k, l$. Notice then if we color i, j, k, l by a, b, c, d, respectively, in R, the so obtained coloring f is monotone on its domain and the resulting colored poset does not contain a tight X. Thus f extends to R. Then $f|_S \in \mathbf{D}$. Since ρ is reflexive, if we color the vertices of both S and S' accordingly to $f|_S$ in R, we obtain an extendible colored quasiordered set. On the other hand i^*, j^*, r^*, k^*, l^* form a tight X in this colored quasiordered set, a contradiction. The existence of r^+ such that $i', j' \leq r^+ \leq k', l'$ is obtained similarly. $\square$

A subset $\{i_0, \ldots, i_l, i'_0, \ldots, i'_l\}$ of a representation $(R, S \cup S')$ is called an *l-step tilted ladder* if $\{i_0, \ldots, i_l\} \subseteq S$, $\{i'_0, \ldots, i'_l\} \subseteq S'$, $i_0 \leq \cdots \leq i_l$, $i'_0 \leq \cdots \leq i'_l$ and either $i_v \leq i'_{v+1}$, $0 \leq v \leq l-1$ or $i_{v+1} \geq i'_v$, $0 \leq v \leq l-1$. See Figure 3. A tilted ladder $\{i_0, \ldots, i_l, i'_0, \ldots, i'_l\}$ with $i_v \leq i'_{v+1}$, $0 \leq v \leq l-1$ ($i_{v+1} \geq i'_v$, $0 \leq v \leq l-1$) is *non-crossed* if $i'_0 \not\leq i_l$ ($i_0 \not\leq i'_l$) in R.

Now we have all the tools at our disposal to state and prove our main theorem.

Theorem 3.4. *Let P be the 6-element poset in Figure 1,* $\mathbf{P}$ *an order primal algebra of finite type related to P, and Σ a finite basis for the variety $\mathscr{V}_P$ generated by* $\mathbf{P}$*. Let* $\mathbf{B}$ *be an algebra that supports a majority term operation m and term operations satisfying Σ. Let ρ be a reflexive compatible binary relation of* $\mathbf{B}$*. Then every edge of ρ is in a directed cycle of length 5 in the digraph (B, ρ). Hence every variety that supports a majority term m and terms satisfying Σ is 5-permutable.*

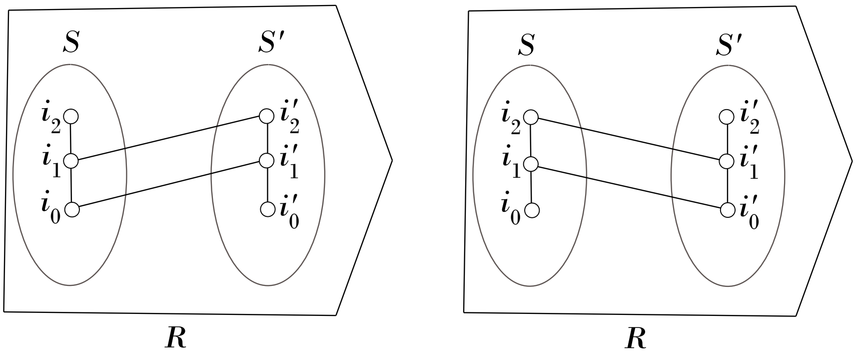


FIGURE 3. Two-step tilted ladders in representation $(R, S \cup S')$

Proof. The last statement of the theorem immediately follows from the first statement of the theorem by Theorem 3.1 of Hagemann and Mitchke. So we prove the first statement.

Let us assume to the contrary that $(Y_0, Z_0) \in \rho$ and there is no directed path of length 4 from Z_0 to Y_0 in the digraph (B, ρ). Our aim is to get a contradiction. We mention that if B was finite, we could basically describe B by a (finite) representation, and we could give a much simpler argument than what follows. However, in general, B may be infinite, not even locally finite, which causes technical difficulties in the proof. As we earlier remarked, we may assume without loss of generality that B is a subuniverse of $\mathbf{P}^H$ for some set H.

We define two subalgebras of $\mathbf{P}^H$. Let $\mathbf{C}_0$ equal the subalgebra generated by $\{Y_0, Z_0\}$ in $\mathbf{P}^H$, and let $\mathbf{D}_0$ equal the subalgebra generated by

$$\{m(W_1, W_2, W_3) \mid W_i \in C_0,\ i = 1, 2, 3\}$$

in $\mathbf{P}^H$. We note that

$$\{Y_0, Z_0\} \subseteq C_0 \subseteq D_0 \subseteq B,$$

and both $\mathbf{C}_0$ and $\mathbf{D}_0$ are finite. Indeed, $\mathbf{D}_0$ is a finitely generated algebra in the variety generated by a finite algebra, namely by $\mathbf{P}$. Since $D_0 \subseteq P^H$ is finite, there exist finitely many elements in H, say n of them, such that the projection map from D_0 to those coordinates is bijective. Let Y, Z, $\mathbf{D}$, $\mathbf{C}$, $\rho_{\mathbf{D}}$ be the images of Y_0, Z_0, $\mathbf{D}_0$, $\mathbf{C}_0$ and $\rho|_{\mathbf{D}_0}$ under such a bijective projection. Let $\rho_{\mathbf{C}} = \rho_{\mathbf{D}}|_C$. Since the bijective projection from $\mathbf{D}_0$ to $\mathbf{D}$ is an isomorphism of algebras, there is no directed path of length 4 from Z to Y in the digraph $(D, \rho_{\mathbf{D}})$.

Let $(T, S \cup S')$ and $(R, S \cup S')$ be representations for ρ_C and ρ_D, respectively, where

$$S = \{s_1, \dots, s_n\} \text{ and } S' = \{s'_1, \dots, s'_n\}.$$

Then the map $i \mapsto i'$ is an isomorphism between the restrictions of T to S and S', and also between the restrictions of R to S and S'. As we remarked earlier, we may assume that $R \subseteq T$.

We define two colored digraphs related to $\mathbf{C}$ and $\mathbf{D}$, respectively. First, we define the colored digraph $W_{\mathbf{C}}$ for $\mathbf{C}$. The base digraph of $W_{\mathbf{C}}$ is obtained from four copies

$$(T_0, S_0 \cup S'_0), \dots, (T_3, S_3 \cup S'_3)$$

of $(T, S \cup S')$ by the natural identifications

$$S'_0 = S_1,\ S'_1 = S_2,\ S'_2 = S_3$$

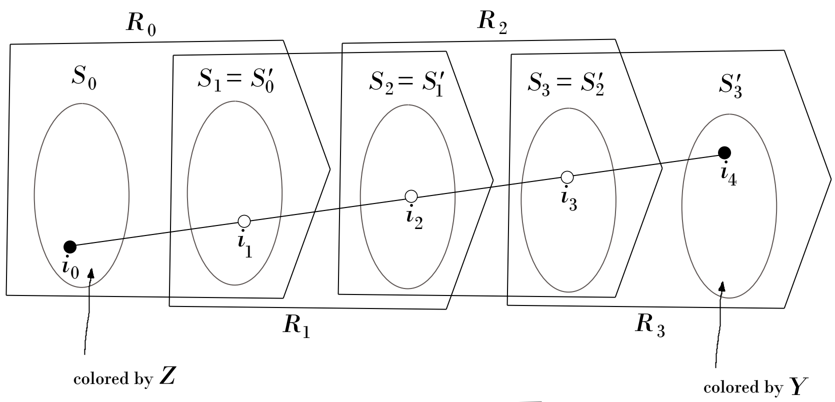
where natural identification is meant to identify the relevant copies of s' and s for each $s \in S$. The coloring of $W_{\mathbf{C}}$ is obtained by coloring the elements of S_0 by Z and coloring the elements of S_3' by Y. The colored digraph $W_{\mathbf{D}}$ is defined similarly by using four copies

$$(R_0, S_0 \cup S_0'), \ldots, (R_3, S_3 \cup S_3')$$

of the representation $(R, S \cup S')$ of $\rho_{\mathbf{D}}$. Observe that neither $W_{\mathbf{C}}$ nor $W_{\mathbf{D}}$ is extendible, for otherwise (Y, Z) would lie in a 5-cycle of $(C, \rho_{\mathbf{C}})$ or $(D, \rho_{\mathbf{D}})$ and hence (Y_0, Z_0) would lie in a 5-cycle of (B, ρ).

Let $\hat{W}_{\mathbf{D}}$ be the colored quasiordered set obtained from $W_{\mathbf{D}}$ by taking the transitive closure of the relation of $W_{\mathbf{D}}$ while preserving the coloring of $W_{\mathbf{D}}$. Then $\hat{W}_{\mathbf{D}}$ is a non-extendible colored quasiordered set. So it contains a twisted edge or a tight X. Now, our aim is to prove the following.

Claim: *There is a non-crossed two-step tilted ladder that is contained in both of the representations $(T, S \cup S')$ and $(R, S \cup S')$.*



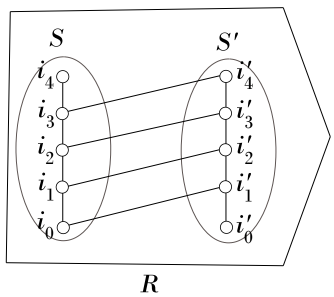


FIGURE 4. A twisted path in $W_{\mathbf{D}}$ coming from a twisted edge in $\hat{W}_{\mathbf{D}}$ and the corresponding four-step tilted ladder in R. The vertices of the path in the upper picture are labeled by the corresponding elements in S. The black nodes are colored vertices. The one labeled by i_0 is colored by q and the one labeled by i_4 is colored by p where $q \not\leq p$.

First, let us look at the case, when $\hat{W}_{\mathbf{D}}$ contains a twisted edge. Then $W_{\mathbf{D}}$ must contain a directed path of length 4, say

$$u_0 \to u_1 \to u_2 \to u_3 \to u_4$$

where $u_j \in S_j$ for $0 \leq j \leq 3$, $u_4 \in S_3'$, u_0 is colored by q and u_4 is colored by p such that $q \not\leq p$. We call this path a *twisted path*. Suppose that u_j is a copy of $i_j \in S$ if $0 \leq j \leq 4$. We depicted the situation in Figure 4.

Clearly, the edges in the twisted path yield that $i_j \leq i'_{j+1}$, $0 \leq j \leq 3$ in R. Hence, by item (1) of Lemma 3.3, we also have that $i_0 \leq \cdots \leq i_4$ in S and $i'_0 \leq \cdots \leq i'_4$ in S'. So the elements i_j and i'_j where $0 \leq j \leq 4$ constitute a four-step tilted ladder in R. Since $R \subseteq T$, the same four-step tilted ladder is contained in T as well.

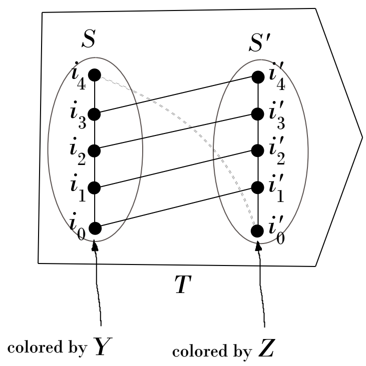


FIGURE 5. The four-step tilted ladder in $(T, Y \cup Z)$ with the cross-edge (i'_0, i_4) where i_4 is colored by p and i'_0 is colored by q.

We shall prove that this ladder is non-crossed, that is, $i'_0 \not\leq i_4$ in T. Suppose to the contrary that $i'_0 \leq i_4$ in T. Since $(Y,Z) \in \rho_{\mathbf{C}}$, the colored quasiordered set $(T, Y \cup Z)$ where Y and Z are considered as partial maps with domains S and S', respectively, is extendible, see Figure 5. It follows that $Z(i'_0) \leq Y(i_4)$ which contradicts the fact that

$$Z(i'_0) = q \not\leq p = Y(i_4).$$

Thus $(T, S \cup S')$ contains a non-crossed four-step tilted ladder. Observe then that $i_0, i_1, i_2, i'_0, i'_1$ and i'_2 form a non-crossed two-step tilted ladder in $(T, S \cup S')$. Moreover, this is a two-step tilted ladder also contained and non-crossed in $(R, S \cup S')$.

In the case when $\hat{W}_{\mathbf{D}}$ contains a tight X, our aim is to prove the same conclusion, namely, that there is a non-crossed two-step tilted ladder contained in both of $(T, S \cup S')$ and $(R, S \cup S')$. The 4 colored elements of the tight X fall in the union of S_0 and S'_3. So either one of S_0 and S'_3 contains 1 colored element and the other contains 3 of them, or both of S_0 and S'_3 contain 2 colored elements. In either case, the midpoint of the tight X is in R_t for some $0 \leq t \leq 3$ and is denoted by r_t. Then in $W_{\mathbf{D}}$, r_t is connected to the colored elements of the tight X via four directed paths whose vertices—possibly apart from r_t—are in the union of the S_v, $0 \leq v \leq 3$, and S'_3, and each of these paths has at most one vertex in each of the S_v, $0 \leq v \leq 3$, and S'_3. We call the subdigraph formed by these four directed paths and r_t a *tight X of paths* in $W_{\mathbf{D}}$. An example of a tight X of paths is depicted in the first picture of Figure 6.

We call the four directed paths the *branches* of the tight X of paths. We make this definition more precise: if r_t is in S_v for some $0 \leq v \leq 3$ we count r_t to the branches, otherwise the branches are meant without r_t. Clearly, the branches with endpoints in S_0 have the same length. The same is true for the branches with endpoints in S'_3. We say that a branch is long if it has length at least 2. So either all of the branches with endpoints in S_0 are long or all of the branches with endpoints in S'_3 are long. Without loss of generality we assume that the branches with endpoints in

S'_3 are long. By the use of item (1) of Lemma 3.3, similarly as we saw it for twisted paths, each of these long branches induces a tilted ladder of at least two steps in R. However, some of the induced tilted ladders might not be non-crossed.

Suppose that $r_t \in W_{\mathbf{D}}$ is a copy of $r^* \in R$ and the endpoints of the branches in R_t are the copies of i^*, j^*, k^* and l^* in R, respectively, where

$$i^* \in \{i, i'\},\ j^* \in \{j, j'\},\ k^* \in \{k, k'\},\ l^* \in \{l, l'\} \text{ for some } i, j, k, l \in S.$$

Clearly, two of i^*, j^*, k^* and l^* are less than or equal to r^* and two of them are greater than or equal to r^*. We may assume that $i^*,\ j^* \leq r^* \leq k^*,\ l^*$. By applying item (2) of Lemma 3.3, there exists an $r^+ \in R$ such that $i',\ j' \leq r^+ \leq k',\ l'$.

Let $u \in W_{\mathbf{D}}$ be one of the colored elements in S_0. So u is an endpoint of a branch of the tight X of paths. Suppose that u is a copy of $h' \in S'$ and the other endpoint of the branch in R_t is a copy of one of the elements $i',\ j',\ k',\ l'$, say l', in S'. By item (1) of Lemma 3.3, the edges of this branch yield a chain in S'— the S' part of the tilted ladder related to this branch— with maximal element h' and minimal element l' . By $r^+ \leq l'$, we obtain $r^+ \leq h'$. Thus, all of the colored vertices of the tight X of paths in S_0 are copies of elements of S' that are comparable with r^+ in R.

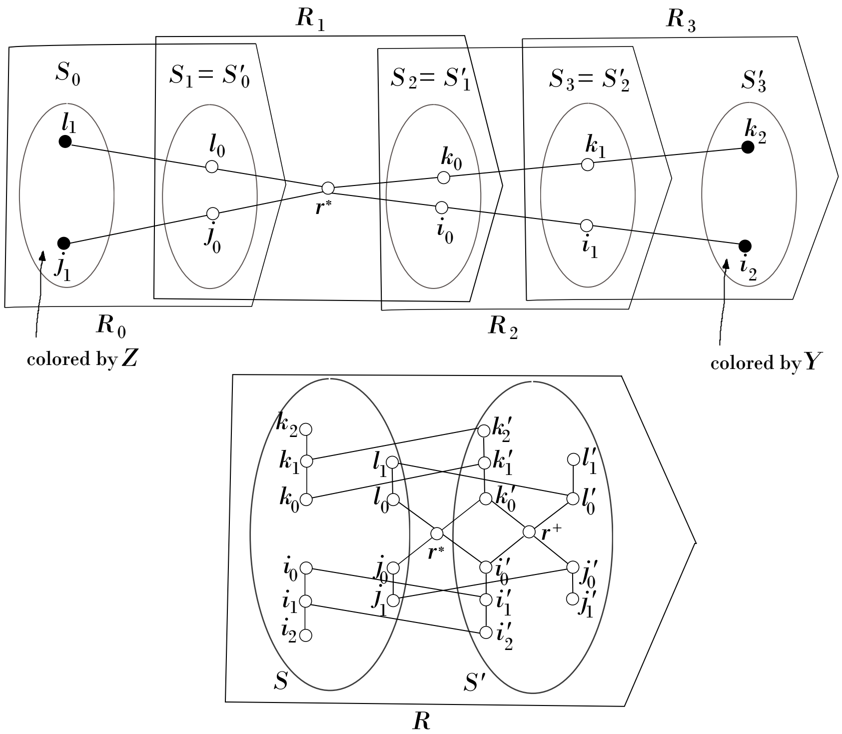


FIGURE 6. A tight X of paths in $W_{\mathbf{D}}$ coming from a tight X in $\hat{W}_{\mathbf{D}}$ and the corresponding edge structure of R. The vertices of the tight X of paths in the upper picture are labeled by the corresponding elements in S and $r^* \in R$. The black nodes are colored vertices. The ones labeled by $i_2,\ j_1,\ k_2,\ l_1$ are colored by $a,\ b,\ c,\ d$, respectively.

Before continuing the main line of our proof, we supply an example in Figure 6 to illustrate the notions introduced in the preceding few paragraphs.The tight X of paths in the first picture of Figure 6 has two long branches of length 2: the paths that are labeled by the i and the k, respectively. The paths that are labeled by the j and the l, respectively, are short branches of length 1. The tight X of paths induces some edges in R as shown in the lower picture of the figure. The edges between S and S' and the edges with endpoint r^* are directly obtained from edges of the tight X of paths. The vertical edges come from the so obtained edges which connect S and S' by the use of item (1) of Lemma 3.3. The vertex r^+ and the edges with endpoint r^+ are obtained from the edges with endpoint labeled by r^* by item (2) of Lemma 3.3. Observe that the four branches of the tight X of paths induce four tilted ladders in the lower picture: the ones given by the i, the j, the k and the l, respectively. By $R \subseteq T$, $(T, S \cup S')$ also contains the edges induced by the tight X of paths in R.

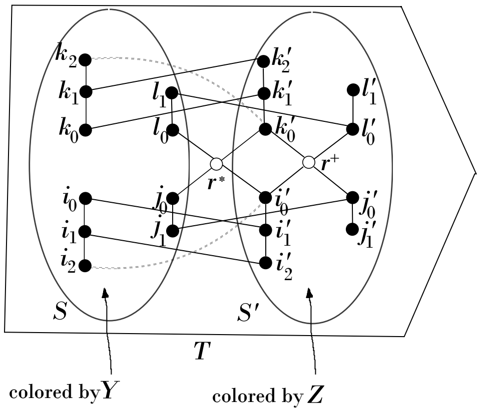


FIGURE 7. The shape of $(T, Y \cup Z)$ with tilted ladders and cross-edges (i_2, i_0') and (k_0', k_2) indicated if $W_{\mathbf{D}}$ contains the tight X of paths in Figure 6. The vertices i_2, j_1', k_2, l_1' are colored by a, b, c, d, respectively.

It turns out that in the example, one of tilted ladders labeled by the i or the k is non-crossed. Suppose to the contrary that $i_2 \leq i_0'$ and $k_0' \leq k_2$, see Figure 7. Hence

$$i_2,\ j_1' \leq r^+ \leq k_2,\ l_1'.$$

Moreover, because of $(Y, Z) \in \rho_{\mathbf{C}}$, the colored quasiordered set $(T, Y \cup Z)$ is extendible. This is impossible, since i_2, j_1', k_2, l_1' are colored by a, b, c, d, respectively, in $(T, Y \cup Z)$ and so i_2, j_1', r^+, k_2, l_1' would constitute a tight X in $(T, Y \cup Z)$.

We return to the proof of the general case. By $R \subseteq T$, the edges induced by the tight X of paths in R are also contained in T. We prove that at least one of the tilted ladders corresponding to a long branch ending in S_3' is non-crossed. Suppose that all of them were crossed.

Let $u \in W_{\mathbf{D}}$ be one of the colored endpoints of a branch in S_3'. Suppose that u is a copy of $h \in S$ and the other endpoint of the branch is a copy of one of the elements i', j', k', l', say k', in S'. Since the tilted ladder corresponding to this long branch is crossed, $k' \leq h$ in T. By $r^+ \leq k'$, we have $r^+ \leq h$. Thus, all of the colored vertices of the tight X of paths in S_3' are copies of elements in S that are comparable with r^+ in T.

Now we can finish off the proof like in the example. Since $(Y, Z) \in \rho_{\mathbf{C}}$, the colored quasiordered set $(T, Y \cup Z)$ is extendible. The four colored elements of the tight X of paths are copies of four elements colored by a, b, c, d, respectively, $(T, Y \cup Z)$. We already verified above that these four colored elements of $(T, Y \cup Z)$ are comparable with r^+. Actually, the proof of this fact also gives that these five elements constitute an X. This X clearly is a tight X in $(T, Y \cup Z)$, a contradiction.

Hence there is a non-crossed tilted ladder in T which corresponds to a long branch, so this tilted ladder must have at least two-steps. This implies, just as we saw it in the case of a twisted edge, that there is a non-crossed two-step tilted ladder in $(T, S \cup S')$ and hence in $(R, S \cup S')$. This concludes the proof of the claim.

Thus, the non-extendibility of $W_{\mathbf{D}}$ yields that there is a non-crossed two-step tilted ladder contained in $(R, S \cup S')$ and $(T, S \cup S')$. We shall define four elements Q_a, Q_b, Q_c and Q_d in $\mathbf{C}$ such that $Q_a, Q_b \xrightarrow{\rho_{\mathbf{C}}} Q_c, Q_d$. Let us take a non-crossed two-step tilted ladder in $(T, S \cup S')$ that is determined by the vertices $i \leq j \leq k$ of S. First, we define Q_a as a map from S to P:

$$Q_a(t) := \begin{cases} a, & \text{if } i \leq t \leq k, \\ 1, & \text{if } i \leq t \not\leq k, \\ 0, & \text{otherwise.} \end{cases}$$

The map Q_b from S to P is defined similarly to Q_a by using b instead of a. We repeat the preceding definition of Q_a and Q_b for Q_c and Q_d, except that we use c, d, i', j', k' and S' instead of a, b, i, j, k and S, respectively. In this way we obtain two maps Q_c and Q_d from S' to P. The so defined maps Q_a, Q_b, Q_c and Q_d are clearly monotone partial maps from T to P. Let us color the elements of S and S' by Q_a and Q_c, respectively, in T.

Obviously, the colored quasiordered set $(T, Q_a \cup Q_c)$ contains no tight X. Since $i \leq j \leq k$ determine a non-crossed tilted ladder contained in $(T, S \cup S')$, there is no edge in T from the interval $[i', k']$ to the interval $[i, k]$. Hence $(T, Q_a \cup Q_c)$ contains no twisted edges as well. Thus, the colored quasiordered set $(T, Q_a \cup Q_c)$ is extendible. So $Q_a \xrightarrow{\rho_{\mathbf{C}}} Q_c$, and hence Q_a and Q_c are in $\mathbf{C}$. We obtain $Q_a \xrightarrow{\rho_{\mathbf{C}}} Q_d$ and $Q_b \xrightarrow{\rho_{\mathbf{C}}} Q_c, Q_d$ similarly, hence $Q_b, Q_d \in \mathbf{C}$. Thus $Q_a, Q_b \xrightarrow{\rho_{\mathbf{D}}} Q_c, Q_d$.

For the preimages Q'_a, Q'_b, Q'_c, Q'_d of Q_a, Q_b, Q_c, Q_d in C_0 we have $Q'_a, Q'_b \xrightarrow{\rho} Q'_c, Q'_d$. Moreover the majority operation m is compatible with ρ on B hence $Q'_a, Q'_b \xrightarrow{\rho} m(Q'_a, Q'_b, Q'_c) \xrightarrow{\rho} Q'_c, Q'_d$ where $m(Q'_a, Q'_b, Q'_c)$ is in D_0. Therefore, there is an element $M \in D$ such that $Q_a, Q_b \xrightarrow{\rho_{\mathbf{D}}} M \xrightarrow{\rho_{\mathbf{D}}} Q_c, Q_d$. From this fact we derive a contradiction.

We define a colored digraph (U, f). The digraph U is formed by four copies $(R_v, S_v \cup S'_v)$, $0 \leq v \leq 3$, of $(R, S \cup S')$ with the natural identifications $S'_0 = S'_1 = S_2 = S_3$.The partial map f is defined by coloring the elements in S_0, S_1, S'_2 and S'_3 by Q_a, Q_b, Q_c and Q_d, respectively. The existence of M guarantees that (U, f) is extendible. On the other hand, the two-step tilted ladder $i \leq j \leq k$, $i' \leq j' \leq k'$ in $(R, S \cup S')$ from the definition of Q_a has a copy $i_v \leq j_v \leq k_v$, $i'_v \leq j'_v \leq k'_v$ in $(R_v, S_v \cup S'_v)$ for each $0 \leq v \leq 3$. We sketched the situation in Figure 8. Notice then that $i_0, i_1 \leq j'_0 \leq k'_2, k'_3$ form a non-extendible colored digraph contained in (U, f) where i_0, i_1, k'_2, k'_3 are colored by a, b, c, d respectively, a contradiction. $\square$

As we explained at the beginning of this section, Theorem 3.4 yields the following consequence.

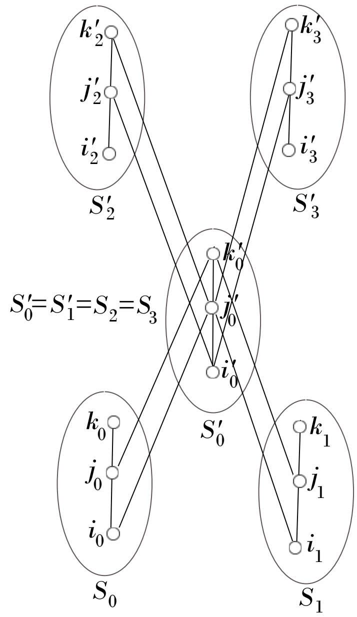


FIGURE 8. A part of digraph U with two-step tilted ladders. The vertices $i_0, i_1, j_0', k_2', k_3'$ form a non-extendible colored digraph in (U, f).

Corollary 3.5. *For any $n \geq 5$, n-permutability is not join-prime in the lattice of interpretability types of varieties.*

We do not know if the corollary holds when $n = 3$ or $n = 4$. Next we are going to prove that 4-permutability cannot be achieved in Theorem 3.4. In order to do this we need to define the notion of a G-obstruction for a digraph G. A *G-colored digraph* is a pair (H, f) where H is a digraph and f is a partial map from H to G. A G-colored digraph is *extendible* if there is an edge-preserving total map from H to G that extends f. A G-colored digraph (H, f) is a *G-obstruction* if (H, f) is non-extendible but any (H', f') properly contained in (H, f) is extendible. It is immediate from the definition that if G is reflexive, then the base digraph of any G-obstruction is a connected irreflexive digraph. We make use of the following fact, cf. Theorem 3.8 in [13], in the proof of the next proposition: a digraph G admits a majority operation if and only if the number of colored elements in any G-obstruction is at most 2.

Proposition 3.6. *Let P be the 6-element poset in Figure 1, $\mathbf{P}$ an order primal algebra of finite type related to P, and Σ a finite basis for the variety $\mathcal{V}_P$ generated by $\mathbf{P}$. Let $\mathbf{B} \leq \mathbf{P}^7$ be the subalgebra defined by the representation (Q, S) in Figure 9 and ρ the reflexive binary relation on B defined by the representation $(R, S \cup S')$ in Figure 9. Then the digraph (B, ρ) admits a majority operation m and operations satisfying Σ. Moreover, $((a,b,c,c,c,1,1),(a,b,d,1,1,d,d))$ is an edge of (B, ρ), but no directed cycle of length 4 contains this edge in (B, ρ). Hence there exists a variety that is not 4-permutable and supports a majority term m and terms satisfying Σ.*

Proof. Let $\tilde{B}$ denote the digraph (B, ρ). It should be clear that ρ is a reflexive relation on B and that the digraph $\tilde{B}$ admits operations satisfying Σ. Let

$$f_c = (a,b,c,c,c,1,1) \text{ and } f_d = (a,b,d,1,1,d,d).$$

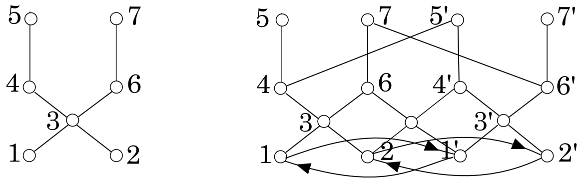


FIGURE 9. A representation (Q,S) of $\mathbf{B} \leq \mathbf{P}^7$ and a representation $(R, S\cup S')$ of the binary relation ρ on B where $S = \{1,2,3,4,5,6,7\}$ and $S' = \{1',2',3',4',5',6',7'\}$.

It is obvious now that $(f_c, f_d) \in \rho$. In order to prove that there is no directed path of length 3 from f_d to f_c in $\tilde{B}$ we define a colored digraph $U_{\mathbf{B}}$. We put together $U_{\mathbf{B}}$ from three separate copies $(R_j, S_j \cup S'_j)$, $0 \leq j \leq 2$, of $(R, S\cup S')$ with the natural identifications $S'_0 = S_1, S'_1 = S_2$ and with coloring the elements of S_0 by the components of f_d and the elements of S'_2 by the components of f_c in an orderly manner. Let $\hat{U}_{\mathbf{B}}$ be the colored quasiordered set obtained from $U_{\mathbf{B}}$ by taking the transitive closure of the relation of $U_{\mathbf{B}}$. Observe now that there is a directed path of length 3 from f_d to f_c in (B,ρ) if and only if $U_{\mathbf{B}}$ is extendible if and only if $\hat{U}_{\mathbf{B}}$ is extendible. Nevertheless, the latter condition does not hold since—with the notation $S_j = \{1_j, \dots, 7_j\}$ and $S'_j = \{1'_j, \dots, 7'_j\}$ for $0 \leq j \leq 2$—the colored vertices $1_0, 2_0, 5_0, 7'_2$ and the midpoint (the vertex distinct from the elements of S_1 and S'_1) of R_1 form a tight X in $\hat{U}_{\mathbf{B}}$.

The main part of the proof is to verify that the digraph $\tilde{B}$ admits a majority operation. As we mentioned preceding the present proposition, it suffices to prove that every $\tilde{B}$-obstruction has at most two colored elements. Let us suppose to the contrary that (H, f) is a $\tilde{B}$-obstruction with at least three colored elements. So H is a digraph, and f is a partial map from H to $B \subseteq P^7$. We define a P-colored digraph W as follows. For every edge e in H we take a copy $(R_e, S_e \cup S'_e)$ of the representation $(R, S\cup S')$ in Figure 9. If two edges e and g have a common vertex, we make one of the natural identifications $S_e = S_g$, $S_e = S'_g$, $S'_e = S_g$ and $S'_e = S'_g$ accordingly to the type of the incidence of the edges e and g (for example, we make $S_e = S_g$ if the edges e and g have a common tail-vertex). Finally, to complete the definition of W, for any colored vertex h of (H, f) we color the subdigraph $(R_e, S_e \cup S'_e)$ of W by $f(h)$ at the elements of S_e if h is the tail-vertex of e and at the elements of S'_e if h is the head-vertex of e. This is obviously a consistent definition.

Let $\hat{W}$ be the colored quasiordered set obtained from W by taking the transitive closure of the relation of W. We gather some facts about $\hat{W}$ before continuing the main line of our proof. Let α denote the largest equivalence contained in the quasiorder relation of $\hat{W}$. Let $\hat{W}/\alpha$ be the quotient poset of W by α. Notice that as H is connected, $\hat{W}/\alpha$ has two minimal elements: one of them is the α-block that contains the elements 1_e and $1'_e$ for all e, and the other is the α-block that contains the elements 2_e and $2'_e$ for all e. The non-minimal α-blocks are of one element. The quotient poset $\hat{W}/\alpha$ has height 3. Each of the minimal α-blocks contains colored elements, and all of those colored elements are colored by the same color since $|H| \geq 3$.

We return to the main line of our proof. Since (H, f) is non-extendible, W and $\hat{W}$ are non-extendible. So $\hat{W}$ contains an obstruction O in $\hat{W}$. By Lemma 3.2, O is a tight X or a twisted edge. If O is a tight X, then the middle element of O is certainly not a maximal element of $\hat{W}$, i.e., it is not labeled by $5_e, 5'_e, 7_e$ and $7'_e$. Suppose that 3_e is one of the minimal elements of O for some e. Then the middle element of O must be 4_e or 6_e, a contradiction, since both vertices are colored. Similarly, $3'_e$ is not a minimal element in O. Hence the minimal elements of O must fall in the minimal α-blocks of $\hat{W}$, separate ones of course. We say that $h \in \operatorname{Dom}(f)$ entails a colored element $v \in W$ if there is an edge e incident with h such that v is in S_e or S'_e according if h is a head or tail vertex of e. Clearly, (H, f) has two colored vertices that entail the two maximal elements of O. Notice that these two colored vertices of H also entail the minimal elements of O, since these minimal elements are in the two minimal α-blocks of $\hat{W}$. If O is a twisted edge, then (H, f) clearly has two colored vertices that entail the two colored elements of O.

We are on the way to derive a contradiction to our assumption that (H, f) has at least three colored elements. We just proved in the preceding paragraph that there are two colored elements h_1 and h_2 of (H, f) that entail all of the colored elements of O. Let $h \in H$ be a colored element that differs from h_1 and h_2. Let us remove h from (H, f). We prove that the remaining $\tilde{B}$-colored digraph (H', f') is still non-extendible. We remark that in H' and H we have the same edges incident with h_i for each $1 \leq i \leq 2$, since in (H, f) there are no edges between colored elements. Let W' be the colored digraph obtained from W by removing the copies of $(R_e, S_e \cup S'_e)$ for the edges e incident with h. Let $\hat{W}'$ be the relational structure obtained from W' by taking the transitive closure of the relation of W'. Notice that $\hat{W}'$ contains O. So $\hat{W}'$ is non-extendible, and hence (H', f') is non-extendible. This contradicts the minimality of (H, f). Therefore, any $\tilde{B}$-obstruction has two colored elements. Thus $\tilde{B}$ admits a majority operation.

Finally, the variety generated by an algebra whose underlying set is B and whose basic operations are a majority operation preserving ρ and operations satisfying Σ—we have just proved that there exists such an algebra—is not 4-permutable by Theorem 3.1 of Hagemann and Mitshke. □

In Problem 1.6 of [14] Opršal asked whether for any given $\boldsymbol{n} \geq 3$, $\boldsymbol{n}$-permutability is join-prime in the lattice of interpretability types of idempotent varieties. For $\boldsymbol{n} = 2$, the question is settled in the positive by the 2-cube term result of Kearnes and Szendrei in [10]. We thought that, by transforming the proof of Theorem 3.4 for the idempotent case, we would be able prove that for any $\boldsymbol{n} \geq 5$ the answer to Opršal's question is negative, but our attempt to achieve such a result has been unsuccessful so far.

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