QUASIORDER LATTICES OF VARIETIES

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Abstract. The set Quo(**A**) of compatible quasiorders (reflexive and transitive relations) of an algebra **A** forms a lattice under inclusion, and the lattice Con(**A**) of congruences of **A** is a sublattice of Quo(**A**). We study how the shape of congruence lattices of algebras in a variety determine the shape of quasiorder lattices in the variety. In particular, we prove that a locally finite variety is congruence distributive [modular] if and only if it is quasiorder distributive [modular]. We show that the same property does not hold for meet semi-distributivity. From tame congruence theory we know that locally finite congruence meet semi-distributive varieties are characterized by having no sublattice of congruence lattices isomorphic to the lattice $\mathbf{M}_3$. We prove that the same holds for quasiorder lattices of finite algebras in arbitrary congruence meet semi-distributive varieties, but does not hold for quasiorder lattices of infinite algebras even in the variety of semilattices.

1. Introduction

A class of universal algebras of the same type forms a variety if it is closed under taking subalgebras, products and homomorphic images. Varieties are precisely the equational classes that is, they can also be described by the equations that hold in all of its members. It is well known that the shape of congruence lattices of algebras in a variety is intimately connected to Maltsev conditions satisfied by the variety [3, 13]. A.I. Mal'cev proved that a variety is congruence permutable if and only if the variety has a ternary term satisfying $p(y, x, x) \approx y$ and $p(x, x, y) \approx y$. B. Jónsson proved that a variety is congruence distributive (that is, the lattice of congruences of algebras in the variety are distributive) if and only of it has Jónsson terms [10] satisfying a package of certain equations. H.-P. Gumm has provided similar characterization of congruence modularity [8]. The shape of congruence lattices also plays a central role in tame congruence theory [9], where congruence conditions are expressed in term of congruence lattices labelled by five types.

In this paper we are attempting to start the rigorous study of the lattice of compatible preorders (reflexive and transitive relations): how their shape is related to that of congruence lattices, and how to adapt techniques developed for tame congruence theory. We note here that we use the term *quasiorder* for compatible preorders (while usually it is used synonymously with preorders) for simplicity's stake. The preorders of a set X and the quasiorder of an algebra **A** both form an algebraic lattice with respect to inclusion, denoted by $\operatorname{Pre} X$ and $\operatorname{Quo} \mathbf{A}$ respectively.

We note that with the natural inversion $\alpha \mapsto \alpha^{-1}$, $\operatorname{Quo} \mathbf{A}$ can be studied as an *involutive lattice*. However, in this paper we are only concerned about its properties as a lattice (while making use of the existence of inversion). We note that the fixed points of the inversion form Con $\mathbf{A}$, which is a sublattice of Quo $\mathbf{A}$.

The study of more general binary relations of algebras has been studied before. Tolerance (compatible reflexive and symmetric relations) relations of algebras were studied in general [4] and in congruence permutable and modular varieties [6, 7]. In [6] G. Czédli and E. K. Horváth proves the analogue of our Theorems 2.6 and 2.10 for tolerances. However, an important distinction between tolerances and quasiorders is that the tolerance lattice generally does not contain the congruence lattice as a sublattice.

Quasiorders of an algebra generalize not only congruences but also compatible partial orders. The study of lattices of quasiorders were mainly studied from the prespective of lattice representation in [14]. In [5] it was proved that algebras with a majority term have distributive quasiorder lattices. In this paper we not only study congruence distributive and modular varieties, but also congruence meet semi-distributive ones that satisfy the congruence quasi-equation

$$\alpha \wedge \gamma = \beta \wedge \gamma \implies \alpha \wedge \gamma = (\alpha \vee \beta) \wedge \gamma.$$

It turns out that even finite semilattices do not have meet semi-distributive quasiorder lattices. However by using some adaptation of techniques from tame congruence theory we can show that quasiorder lattices of finite algebras in arbitrary congruence meet semi-distributive variety do not have $\mathbf{M}_3$ as a sublattice (c.f. [9] Theorem 9.10).

2. Distributivity and modularity

Definition 2.1. A sequence $p_1, \dots, p_n$ of ternary terms is called *directed Jónsson terms* if they satisfy the identities

$$\begin{aligned} x &= p_1(x,x,y), \\ p_i(x,y,y) &= p_{i+1}(x,x,y) \text{ for } i = 1,\dots,n-1, \\ p_n(x,y,y) &= y, \text{ and} \\ p_i(x,y,x) &= x \text{ for } i = 1,\dots,n. \end{aligned}$$

It is already known that if locally finite variety has Jónsson terms, then it also has directed Jónsson terms (M. Kozik), and later this implication was also proved for general varieties [11]. However, we present a quick proof of this fact using a result of L. Barto [1].

Lemma 2.2. *If a locally finite variety $\mathcal{V}$ has Jónsson terms, then it has directed Jónsson terms.*

Proof. Let $\mathbf{F}_2$ be the free algebra in $\mathcal{V}$ freely generated by x and y, and let $\mathbf{R} = \mathrm{Sg}_{\mathbf{F}_2^3}((x,x,x),(x,y,y),(y,x,y))$ be the generated subalgebra of $\mathbf{F}_2^3$. Consider the relational structure $\mathbb{S} = (F_2; R)$ on the underlying set of $\mathbf{F}_2$ with the ternary relation R. Clearly, this finite relational structure has Jónsson polymorphisms (the Jónsson term operations are compatible with R), so by the result of L. Barto [1], $\mathbb{S}$ has a compatible near-unanimity polymorphism t of arity n. It is important to realize that t is not necessarily a term operation of $\mathbf{F}_2$, it is just a near-unanimity operation that happens to preserve the relation R.

Now consider the tuples

$$(a_i, b_i, c_i) = t((y,x,y),\dots,(y,x,y),(x,y,y),(x,x,x),\dots,(x,x,x))$$

for $i = 1,\dots,n$, where the argument (x,y,y) appears on the i-th coordinate. Since t is a near-unanimity operation on F_2, we have $b_i = t(x,\dots,x,y,x,\dots,x) = x$ for all i, $a_1 = x$, and $c_n = y$. On the other hand, $c_i = t(y,\dots,y,y,x,\dots,x) = a_{i+1}$ for $i = 1,\dots,n-1$.

Since t preserves the relation R, we have $(a_i, b_i, c_i) \in R$, therefore there are ternary terms $p_1,\dots,p_n$ in $\mathcal{V}$ so that

$$(a_i, b_i, c_i) = p_i((x,x,x),(x,y,y),(y,x,y)).$$

Thus $p_i(x,y,x) = b_i = x$, $p_1(x,x,y) = a_1 = x$, $p_n(x,y,y) = c_n = y$ and $p_i(x,y,y) = c_i = a_{i+1} = p_{i+1}(x,x,y)$ for $i = 1,\dots,n-1$. These equalities hold in $\mathbf{F}_2$, which proves that $p_1,\dots,p_n$ is a sequence of directed Jóbsson terms of $\mathcal{V}$. □

Definition 2.3. For an arbitrary $\alpha \in \operatorname{Pre} X$, α^* denotes the equivalance $\alpha \cap \alpha^{-1}$.

Naturally, if α is a quasiorder of $\mathbf{A}$ then α^* is a congruence, and α/α^* is a compatible poset on $\mathbf{A}/\alpha^*$.

Lemma 2.4. *If a finite algebra has directed Jónsson terms, then the lattice of its quasiorders is distributive.*

Proof. Let $\mathbf{A}$ be a finite algebra and $\alpha,\beta,\gamma \in \operatorname{Quo}(\mathbf{A})$ be three of its quasiorders. It is enough to show that

$$(\alpha \vee \beta) \wedge \gamma \le (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$$

as this inequality implies distributivity [3]. Take a pair (a,b) of elements from the left hand side, so $(a,b) \in \gamma$ and there is a sequence $a = a_1, a_2, \dots, a_n = b$ of elements such that $(a_i, a_{i+1}) \in \alpha \cup \beta$. For any of the directed Jónsson terms p_i we have

$$p_i(a,a,b) = p_i(a,a_1,b) \xrightarrow{\alpha\cup\beta} p_i(a,a_2,b) \xrightarrow{\alpha\cup\beta} \cdots$$
$$\cdots \xrightarrow{\alpha\cup\beta} p_i(a,a_{n-1},b) \xrightarrow{\alpha\cup\beta} p_i(a,a_n,b) = p_i(a,b,b)$$

with $a = p_i(a,a_j,a) \xrightarrow{\gamma} p_i(a,a_j,b) \xrightarrow{\gamma} p_i(b,a_j,b) = b$ for all $j = 1,\dots,n$. By using the equations of the directed Jónsson terms we can combine the above paths between $p_i(a,a,b)$ and $p_i(a,b,b)$ into a single path

$$a = c_1 \xrightarrow{\alpha\cup\beta} c_2 \xrightarrow{\alpha\cup\beta} \cdots \xrightarrow{\alpha\cup\beta} c_m = b$$

where $a \xrightarrow{\gamma} c_i \xrightarrow{\gamma} b$ for $i = 1,\dots,m$.

Now we are ready to prove the lemma by induction on the number of γ^* congruence classes between a/γ^* and b/γ^* in the poset γ/γ^*. If $a/\gamma^* = b/\gamma^*$, that is, $(b,a) \in \gamma$, then

$$c_i \xrightarrow{\gamma} b \xrightarrow{\gamma} a \xrightarrow{\gamma} c_{i+1}$$

so $(c_i, c_{i+1}) \in (\alpha\cup\beta)\cap\gamma = (\alpha\cap\gamma)\cup(\beta\cap\gamma)$, which proves that $(a,b) \in (\alpha\wedge\gamma)\vee(\beta\wedge\gamma)$.

If $a/\gamma^* \ne b/\gamma^*$, then let k be the smallest index such that $a/\gamma^* \ne c_k/\gamma^*$. Then we have

$$a = c_1 \xrightarrow{(\alpha\cap\gamma)\cup(\beta\cap\gamma)} c_2 \xrightarrow{(\alpha\cap\gamma)\cup(\beta\cap\gamma)} \cdots \xrightarrow{(\alpha\cap\gamma)\cup(\beta\cap\gamma)} c_{k-1}$$

because $a/\gamma^* = c_1/\gamma^* = \cdots = c_{k-1}/\gamma^*$. On the other hand, $c_{k-1} \xrightarrow{\gamma} a \xrightarrow{\gamma} c_k$, so we also have

$$c_{k-1} \xrightarrow{(\alpha\cap\gamma)\cup(\beta\cap\gamma)} c_k.$$

Therefore, we have $(a, c_k) \in (\alpha \cap \gamma) \vee (\beta \cap \gamma)$ as well. Since $c_k \xrightarrow{\gamma} b$ we can apply the induction hypothesis for the sequence of elements

$$c_k \xrightarrow{\alpha\cup\beta} c_{k+1} \xrightarrow{\alpha\cup\beta} \cdots \xrightarrow{\alpha\cup\beta} c_m = b$$

to get that $(c_k, b) \in (\alpha \cap \gamma) \cup (\beta \cap \gamma)$. This implies $(a, b) \in (\alpha \cap \gamma) \cup (\beta \cap \gamma)$, which concludes the proof of the lemma. □

Actually we have proved more than we stated. We could replace $\alpha \cup \beta$ with any reflexive relation ϱ of $\mathbf{A}$ and the whole proof would go through. Therefore, we have the following proposition.

Proposition 2.5. *Let $\mathbf{A}$ be a finite algebra with directed Jónsson terms, and let ϱ and σ be two reflexive relations of $\mathbf{A}$. Then $\overline{\varrho} \cap \overline{\sigma} = \overline{\varrho \cap \overline{\sigma}}$ where the line over a reflexive relation means the transitive closure.* □

Theorem 2.6. *For any locally finite variety $\mathcal{V}$ the following are equivalent:*
1. *$\mathcal{V}$ is congruence distributive,*
2. *$\mathcal{V}$ has Jónsson terms,*
3. *$\mathcal{V}$ has directed Jónsson terms, and*
4. *$\mathcal{V}$ has distributive quasiorder lattices.*

Proof. (1) ⇒ (2) is well known [10], and (2) ⇒ (3) is proved by Lemma 2.2. For finite algebras Lemma 2.4 proves (3) ⇒ (4). In particular, the finitely generated free algebras in the variety have distributive quasiorder lattices, which in a natural way can be used to show that all algebras in the variety have distributive quasiorder lattices. Finally, (4) ⇒ (1) holds trivially as congruence lattices are sublattices of quasiorder lattices. □

We will now prove the analogue of the above theorem for congruence modular varieties.

Definition 2.7. A sequence $p_1, \dots, p_n, q$ of ternary terms is called *directed Gumm terms* if they satisfy the identities

$$\begin{aligned} x &= p_1(x, x, y), \\ p_i(x, y, y) &= p_{i+1}(x, x, y) \text{ for } i = 1, \dots, n-1, \\ p_n(x, y, y) &= q(x, y, y), \\ q(x, x, y) &= y, \text{ and} \\ p_i(x, y, x) &= x \text{ for } i = 1, \dots, n. \end{aligned}$$

Lemma 2.8. *If a locally finite variety $\mathcal{V}$ has Gumm terms, then it has directed Gumm terms.* □

We won't repeat the details, but this can be proved the same way as Lemma 2.2 by using an edge term instead of a near-unanimity one and the main result in [2]. Also, the lemma was proved for general varieties in [11].

Theorem 2.9. *If* $\mathbf{A}$ *is a finite algebra admitting directed Gumm terms, then its lattice of quasiorders is modular.*

Proof. We have to prove that for compatible preorders α, β and γ, $\alpha \subseteq \gamma$ implies $(\alpha\vee\beta)\wedge\gamma \subseteq \alpha\vee(\beta\wedge\gamma)$. Assume that there is a pair (a,b) which is an element of the left side but not an element of the right. There are elements $c_0 = a, c_1, \dots, c_n = b$ such that for all i $(c_i, c_{i+1}) \in \alpha\cup\beta$. Choose this counterexample in a way that the "height difference" between a and b in γ (meaning the maximal l such that there are $a = d_0, d_1, \dots, d_l = b \in A$ so that for all i $(d_i, d_{i+1}) \in \gamma\backslash\gamma^{-1}$) is minimal. Now take the α-β chain of the previous theorem:

$$a = p_1(a,a,b) \xrightarrow{\alpha\cup\beta} p_1(a,c_1,b) \xrightarrow{\alpha\cup\beta} \dots \xrightarrow{\alpha\cup\beta} p_1(a,b,b) = p_2(a,a,b)$$
$$\xrightarrow{\alpha\cup\beta} \dots \xrightarrow{\alpha\cup\beta} p_k(a,b,b) = q(a,b,b) \xrightarrow{\alpha\cup\beta} \dots \xrightarrow{\alpha\cup\beta} q(b,b,b) = b.$$

Every element of this chain up until $q(a,b,b)$ is between a and b in γ (as $a = p_i(a,c_j,a) \xrightarrow{\gamma} p_i(a,c_j,b) \xrightarrow{\gamma} p_i(b,c_j,b) = b$), and from each of them there obviously is an $\alpha\vee\beta$-edge to b. By the minimal height choice of (a,b), each element of the chain between a and $q(a,b,b)$ is either in the same γ^*-class as a, or has an $\alpha\vee(\beta\wedge\gamma)$-edge going from it to b. But this is only possible if $q(a,b,b)$ is of the γ^*-class of a: otherwise take the first element of the chain not in the γ^*-class of a, and denote it with c. Thus there is an $\alpha\vee(\beta\wedge\gamma)$-edge from c to b, but there also is an $\alpha\vee(\beta\wedge\gamma)$-edge from a to c, as the α-β chain from a to c only has edges that are in γ. This contradicts the choice of (a,b). Thus $q(a,b,b) \xrightarrow{\gamma} a$, and as $b = q(a,a,b) \xrightarrow{\gamma} q(a,b,b)$, a and b are in the same γ^*-class. Note that this means $a \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(a,b,b)$, because the above path lies in the same γ^*-block up until $q(a,b,b)$.

More generally, for any pair (d,d') that is in both in $\alpha\vee\beta$ and γ^*,

$$(d, q(d,d',d')) \in \alpha\vee(\beta\wedge\gamma),$$

because either $(d,d') \in \alpha\vee(\beta\wedge\gamma)$, and so

$$d \xrightarrow{\alpha\vee(\beta\wedge\gamma)} d' = q(d,d,d') \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(d,d',d'),$$

or (d,d') is a counterexample of minimal height difference (that is, 0) between d and d', in which case the argument for (a,b) applies for (d,d').

Now we specialise further the choice of the counterexample: we assume that n is minimal among all counterexamples $(a, c_1, \dots, c_{n-1}, b)$ where a, b lie in the same γ^*-class. Notice that the assumption $(a,b) \notin \alpha\vee(\beta\wedge\gamma)$ means that n is at least 2. We will differentiate between the cases whether there is an α-edge on at least one end of the α-β chain of length n between a and b.

Assume first that both $a \xrightarrow{\beta} c_1$ and $c_{n-1} \xrightarrow{\beta} b$. Notice that

$$b = q(b,b,b) = q(q(a,a,b),b,b) \xrightarrow{\gamma} q(q(a,b,b),b,b)$$

while

$$b = q(q(a,c_{n-1},b), q(a,c_{n-1},b), b) \xrightarrow{\beta} q(q(a,b,b), q(c_1,c_{n-1},b), b) \xrightarrow{\alpha\cup\beta}$$
$$q(q(a,b,b), q(c_{n-1},c_{n-1},b), b) = q(q(a,b,b),b,b),$$

and the α-β chain connecting b to $q(q(a,b,b),b,b)$ is of length $n-1$. By the choice of (a,b) this means that $b \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(q(a,b,b),b,b)$.

To arrive at a contradiction, we need to prove the existance of an $\alpha\vee(\beta\wedge\gamma)$-edge from a to b, hence we alter the above edge to have its target be at b:

$$b = q(q(q(a,b,b),b,b),q(q(a,b,b),b,b),b) \xleftarrow{\alpha\vee(\beta\wedge\gamma)} q(q(q(a,b,b),b,b),b,b).$$

It's enough now to show that

$$a \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(q(q(a,b,b),b,b),b,b).$$

We have proved that $a \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(a,b,b)$, so the pair $(q(a,b,b),b)$ must be a counterexample too, with $b \xrightarrow{\gamma} q(a,b,b)$. Now we can, in the same way as we proved $a \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(a,b,b)$, deduce

$$q(a,b,b) \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(q(a,b,b),b,b).$$

Hence $a \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(q(a,b,b),b,b)$, and $(q(q(a,b,b),b,b),b)$ is another counterexample (with $b \xrightarrow{\gamma} q(q(a,b,b),b,b))$, so again repeating the earlier argument, we get

$$q(q(a,b,b),b,b) \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(q(q(a,b,b),b,b),b,b).$$

The first case is done.

Now take the other case, where $a \xrightarrow{\alpha} c_1$ or $c_{n-1} \xrightarrow{\alpha} b$ holds, we will assume the former (the other case is similar).

Notice that as $b = q(a,a,b) \xrightarrow{\gamma} q(a,c_1,b) \xrightarrow{\gamma} q(c_1,c_1,b) = b$, the pair $(q(a,c_1,b),q(a,b,b))$ is in γ^* and $\alpha\vee\beta$, and there is an α-β chain between them of length $n-1$. By the choice of (a,b), this means

$$b = q(a,a,b) \xrightarrow{\alpha} q(a,c_1,b) \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(a,b,b),$$

from which we get

$$q(q(a,b,b),b,b) \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(q(a,b,b),q(a,b,b),b) = b.$$

As in the previous case, we can deduce that $a \xrightarrow{\alpha\vee(\beta\wedge\gamma)} q(q(a,b,b),b,b)$, which yields a contradiction for the second case. □

From this, we can prove the following the same way as we did in the distributive case.

Theorem 2.10. *For any locally finite variety $\mathcal{V}$ the following are equivalent:*

(1) *$\mathcal{V}$ is congruence modular,*
(2) *$\mathcal{V}$ has Gumm terms,*
(3) *$\mathcal{V}$ has directed Gumm terms, and*
(4) *$\mathcal{V}$ has modular quasiorder lattices.* □

Problem 2.11. Is there a direct proof showing that if a (locally finite) variety has distributive [modular] quasiorder lattices, then it has directed Jónsson [Gumm] terms?

Problem 2.12. Are Theorems 2.6 and 2.10 true for arbitrary varieties?

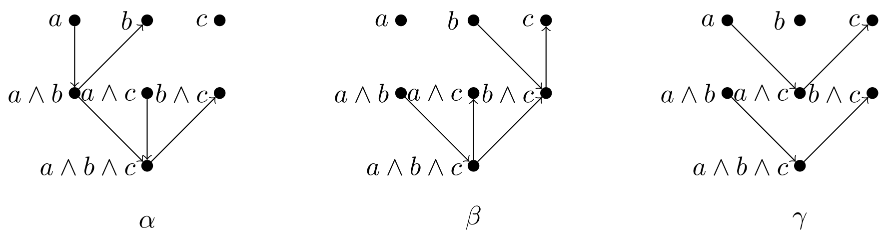

FIGURE 1. The quasiorders α, β, γ of FS(3) satisfying $\alpha \wedge \gamma = \beta \wedge \gamma < (\alpha \vee \beta) \wedge \gamma$

3. CONGRUENCE $\wedge$-SEMI-DISTRIVUTIVE VARIETIES

It is a natural question to ask whether Lemma 2.4 is true if we replace "distributivity" with "semi-distributivity". While we could not answer this question (but leaning towards "yes"), the analogue question to meet semi-distributivity is easily answerable in the negative.

Theorem 3.1. Quo(FS(3)) *(the* 3*-generated free semilattice) is not a meet semi-distributive lattice.*

Proof. Let $X = \{a, b, c\}$. Take α_0, β_0, and γ as the quasiorders of $\mathrm{FS}(X)$ generated respectively by (a, b), (b, c), and (a, c). Both $(\gamma \wedge \alpha_0)\backslash\beta_0$ and $(\gamma \wedge \beta_0)\backslash\alpha_0$ contain only a single edge, namely, $(a \wedge b \wedge c, b \wedge c)$, and $(a \wedge b, a \wedge b \wedge c)$, respectively. Therefore, with $\alpha = \mathrm{Tr}(\alpha_0 \cup \{(a \wedge b, a \wedge b \wedge c)\})$, and $\beta = \mathrm{Tr}(\beta_0 \cup \{(a \wedge b \wedge c, b \wedge c)\})$,

$$\alpha \wedge \gamma = \beta \wedge \gamma < (\alpha \vee \beta) \wedge \gamma,$$

the inequality holding because only the right side contains (a, c). Note that α and β are quasiorders: both $(a \wedge b \wedge c, b \wedge c)$, and $(a \wedge b, a \wedge b \wedge c)$ are mapped into themselves or a loop by any unary polynomial, so $\alpha_0 \cup \{(a \wedge b, a \wedge b \wedge c)\}$ and $\beta_0 \cup \{(a \wedge b \wedge c, b \wedge c)\}$ are compatible reflexive relations, hence their transitive closures are quasiorders. Thus $\mathrm{Quo}(\mathrm{FS}(X))$ fails meet semi-distributivity. □

Corollary 3.2. *Congruence meet semi-distributivity of a variety does not imply that the quasiorder lattices of the finite algebras are meet semi-distributive, not even if the variety is assumed to be locally finite.*

By [9] Theorem 9.10, an equivalent characterization of congruence meet semi-distributivity for locally finite varieties is that congruence lattices in the variety do not contain sublattices isomorphic to $\mathbf{M}_3$. We will prove that this second condition (contrary to meet semi-distributivity itself) yields its analogue for quasiorders – but only for finite algebras of the variety.

We will need tame congruence theory (see [9]). Some concepts (types of quotients, traces, pseudo-meet operations etc.) will only be used for congruences. Others we generalize for quasiorders. We will only give the definitions here for the latter.

Suppose $\mu < \nu$ are quasiorders of a finite algebra $\mathbf{A}$. An $U \subseteq A$ is called a (μ, ν)*-minimal set* if there is a unary polynomial f of $\mathbf{A}$ such that $f(A) = U$, and

$f(\nu) \nsubseteq \mu$, and there is no such f for proper subsets of U. We call $\mathbf{A}$ a (μ,ν)-*minimal algebra* if A is a (μ,ν)-minimal set.

From a (μ,ν)-minimal set $U \subseteq A$ a (μ,ν)-minimal algebra $\mathbf{A}|_U$ can be gained the following way: its underlying set is U, and for every polynomial $p \in \mathrm{Pol}_k(\mathbf{A})$ such that $p(U^k) \subseteq U$ add the restriction of p to U as an operation of $\mathbf{A}|_U$.

Though the definition of a tame pair of congruences, and the properties of minimal algebras for tame pairs (Theorem 2.8. of [9]) can be carried over to quasiorders, we will only need the following properties (Lemma 2.3. and Lemma 2.10. of [9], the proofs there can be applied essentially word-for-word).

Proposition 3.3. *Let $\mathbf{B}$ be a finite algebra and $I[\alpha,\beta]$ an interval in* $\mathrm{Quo}(\mathbf{B})$.

1. *If this interval doesn't have a non-constant meet endomorphism μ satisfying $\eta < \mu(\eta)$ for all $\eta < \beta$, then each (α,β)-minimal set of $\mathbf{B}$ is the range of an idempotent unary polynomial.*
2. *If U is an (α,β)-minimal set such that it is the range of an idempotent unary polynomial e of $\mathbf{A}$ satisfying $e(\beta) \nsubseteq \alpha$, then the mapping from* $\mathrm{Quo}(\mathbf{A})$ *to* $\mathrm{Quo}(\mathbf{A}|_U)$ *that maps each quasiorder of $\mathbf{A}$ to its restriction to U, is a lattice homomorphism.*

Lemma 3.4. *Suppose that $\mathbf{B}$ is a finite algebra generating a congruence meet semi-distributive variety, $e \in \mathrm{Pol}_1(\mathbf{B})$ is an idempotent polynomial. Then $\mathbf{B}|_{e(B)}$ also generates a congruence meet semi-distributive variety.*

Proof. A finite algebra generates a congruence meet semi-distributive variety iff it has a collection of ternary terms satisfying an idempotent Maltsev condition [9]. If $\mathbf{B}$ has such terms t_i $(i = 1 \dots n)$, then $e(t_i) \in \mathrm{Pol}_1 \mathbf{B}$ are terms of $\mathbf{B}|_{e(B)}$ satisfying the same identities. □

Lemma 3.5. *Suppose that $\mathbf{L}$ is a lattice of partial orders of a finite set C (that is, a sublattice of the lattice of preorders of C such that no element of this sublattice contains a double edge). Then $\mathbf{L}$ is join semi-distributive.*

Proof. Suppose that $\alpha \vee \beta = \alpha \vee \gamma$ in $\mathbf{L}$, and suppose that $c_1, c_2 \in C$ such that $c_1 \prec c_2$ in $\alpha \vee \beta$. There are elements of $d_0, \dots, d_k \in C$ such that $c_1 = d_0 \xrightarrow{\alpha} d_1 \xrightarrow{\beta}\xrightarrow{\alpha} \dots \xrightarrow{\beta} d_k = c_2$. All the edges of this path are $\alpha \vee \beta$ edges, and as $\alpha \vee \beta$ contains no double edges, all the elements of the path are in $\{c_1, c_2\}$. This means that (c_1, c_2) is in either α or β. Likewise, it is in either α or γ. Putting the two together we get that $(c_1, c_2) \in \alpha \vee (\beta \wedge \gamma)$.

Hence all the covering edges of $\alpha \vee \beta$ are in $\alpha \vee (\beta \wedge \gamma)$. As the latter is a preorder, and C is finite, the whole of $\alpha \vee \beta$ (the transitive closure of its covering edges) is in $\alpha \vee (\beta \wedge \gamma)$. Thus $\mathbf{L}$ is join semi-distributive. □

The preceding lemma does not stand if we omit the finiteness of C (see Lemma 4.2).

Lemma 3.6. *Suppose that C is a finite set, $\mathbf{L} \leq \mathrm{Pre}\, C$ is a lattice of preorders on C, and $\beta, \beta^{-1} \in L$. Then the mapping $\delta \mapsto \delta \wedge \beta^{-1}$ is a lattice homomorphism from the ideal $(\beta]$ of $\mathbf{L}$ to the ideal $(\beta^*]$.*

Proof. It is obvious that this mapping preserves meets. We need to show that for any $\delta_1, \delta_2 \leq \beta$:

$$(\delta_1 \vee \delta_2) \wedge \beta^{-1} \subseteq (\delta_1 \wedge \beta^{-1}) \vee (\delta_2 \wedge \beta^{-1}).$$

Consider an edge (a, b) from the left hand side. There must be elements of C $a = c_0, c_1, \ldots, c_k = b$ so that for each $0 \leq i < k$, (c_i, c_{i+1}) is in either δ_1 or δ_2. As

$$a = c_0 \xrightarrow{\beta} c_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} c_k = b \xrightarrow{\beta} a,$$

all the c_i are in the same β^*-block, so each (c_i, c_{i+1}) is in either $\delta_1 \wedge \beta^{-1}$ or $\delta_2 \wedge \beta^{-1}$. □

Theorem 3.7. *Let* $\mathbf{A}$ *be a finite algebra in an SD($\wedge$) variety. There exists no sublattice of* $\operatorname{Quo}\mathbf{A}$ *isomorphic to* $\mathbf{M}_3$.

Proof. Suppose the contrary, and choose a counterexample of the smallest cardinality. Denote the sublattice isomorphic to $\mathbf{M}_3$ in $\operatorname{Quo}\mathbf{A}$ with $\mathbf{K}$, its elements with $\alpha, \beta, \gamma_1, \gamma_2, \gamma_3$, with α being the smallest and β the largest element. There can be more than one such sublattice, choose one such that β is minimal, and among these one such that α is maximal.

Note that there cannot be a double edge in α, otherwise factoring out by $\alpha \wedge \alpha^{-1}$ would yield a counterexample with a smaller cardinality. β, on the other hand, must have a double edge, because $\mathbf{K}$ is not a join semi-distributive lattice, thus by Lemma 3.5 it is not a lattice of orderings of a finite set.

Claim 1. *β is a congruence.*

By Lemma 3.6, if we change the elements of K to their respective meets with β^{-1}, we get a sublattice of $\operatorname{Quo}\mathbf{A}$ isomorphic to a homomorphic image of $\mathbf{M}_3$. As $\alpha \wedge \beta^{-1} \neq \beta \wedge \beta^{-1}$ (because β has a double edge, and α does not), and $\mathbf{M}_3$ is a simple lattice, we have obtained an other sublattice of $\operatorname{Quo}(\mathbf{A})$ isomorphic to $\mathbf{M}_3$. From the minimality of β we get $\beta = \beta \wedge \beta^{-1}$, which proves the claim.

Claim 2. *$\mathbf{A}$ is an (α, β)-minimal algebra.*

Note that K satisfies the conditions of the first statement of Lemma 3.3, as the existance of such a μ would yield a sublattice of $\operatorname{Quo}(\mathbf{A})$ with smallest element larger than α (applying it pointwise to the elements of K). Thus if A is not (α, β)-minimal, then it has an (α, β)-minimal set U and an idempotent unary polynomial e such that $e(A) = U$. Now, by the second statement of Lemma 3.3, the restriction of the elements of $\operatorname{Quo}(\mathbf{A})$ to U is a homomorphism from $\operatorname{Quo}(\mathbf{A})$ to $\operatorname{Quo}(\mathbf{A}|_U)$. This homomorphism maps α and β into different quasiorders, thus maps the elements of K to elements of $\operatorname{Quo}(\mathbf{A}|_U)$ forming a sublattice isomorphic to $\mathbf{M}_3$, as $\mathbf{M}_3$ is a simple lattice. If $\mathbf{A}$ was not (α, β)-minimal, $\mathbf{A}|_U$ would be a counterexample of smaller cardinality.

Claim 3. *$\mathbf{A}$ is a $(0, \beta)$-minimal algebra.*

Take a unary polynomial of $\mathbf{A}$ that is not bijective. As $\mathbf{A}$ is (α, β)-minimal, it maps β-edges into α-edges. But β is a congruence, thus all its edges are double edges, and α doesn't have any non-loop double edges. So all non-bijective polynomials must map any β-edge into a loop. Claim 3 is proved.

Take an arbitrary congruence β' such that $\beta' \prec \beta$ in $\operatorname{Con}\mathbf{A}$. Obviously A is also (β',β)-minimal, and as A is in a variety omitting 1 and 2, it has a pseudo-meet binary polynomial p for the covering pair (β',β). Consider the (β',β)-trace of A, this has two β'-classes, one of them has only the element 1 which is neutral for p. The trace cannot have more than 2 elements: for any $a \neq 1$ in the trace the unary polynomials $p(x,a)$ and $p(a,x)$ cannot be bijective (mapping both 1 and a into a), which means by the $(0,\beta)$-minimality of A that for any b in the trace $p(b,a) = p(a,b) = a$ stands. This fact clearly rules out the existence of more than one non-1 element in the trace.

Thus β must have a two element (say: x and y) block. Now (x,y) must be in γ_1 or γ_2 by $\gamma_1 \wedge \gamma_2 = \beta$. Likewise, it must be in either γ_1 or γ_3, and in either γ_2 or γ_3. So (x,y) is an element of at least two of the γ_i-s, and thus $(x,y) \in \alpha$. Similarly, $(y,x) \in \alpha$, which contradicts the fact that α does not have a double edge. $\square$

4. Infinite semilattices

The aim of this section is to show that Theorem 3.7 does not hold if we omit the finiteness of $\mathbf{A}$, even if we add the condition that the variety itself is locally finite. This may be surprising: in the distributive and modular cases, we first proved Theorem 2.4 and Theorem 2.9 for finite algebras, and it was an immediate consequence that it is also true for infinite algebras in locally finite varieties. This difference is due to meet semi-distributivity not being a lattice identity, hence it is possible that an algebra is not quasiorder meet semi-distributive, while all of its finitely generated subalgebras are so.

We begin with giving a counterexample to the infinite generalization of Lemma 3.5.

Example 4.1. Take C as the set of finite ternary fractions in the interval $[0,1]$.

We define on C the preorders $\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3$ by giving a generating set for each (to which we will refer to as their cores). The set of the core edges consists of all the pairs of the form $(\frac{a}{3^j}, \frac{a+1}{3^j})$, where a, j are integers, $0 \leq j$ and $0 \leq a \leq 3^j - 1$. A core edge is called an *up*, a *neutral*, or a *down* edge, depending on whether the remainder of a modulo 3 is 0, 1, or 2.

The allocation of the core edges to exactly one of the $\overline{\gamma}_i$ will be done recursively. The pair $(0,1)$ is in the core of $\overline{\gamma}_1$. Assume that $(\frac{a}{3^j}, \frac{a+1}{3^j})$ is in the core of $\overline{\gamma}_i$. Then $(\frac{3a}{3^{j+1}}, \frac{3a+1}{3^{j+1}})$ and $(\frac{3a+2}{3^{j+1}}, \frac{3a+3}{3^{j+1}})$ will be in the core of $\overline{\gamma}_{i+1}$, while $(\frac{3a+1}{3^{j+1}}, \frac{3a+2}{3^{j+1}})$ will be in the core of $\overline{\gamma}_{i-1}$.

In the above example and henceforth, the index i is meant to be modulo 3.

Lemma 4.2. *In Example 4.1, the sublattice generated by $\overline{\gamma}_1, \overline{\gamma}_2$ and $\overline{\gamma}_3$ in the lattice of preorders of C contains only partial orders, and it is isomorphic to $\mathbf{M}_3$.*

Proof. It is enough to show that the intersection of any two of the $\overline{\gamma}_i$ is 0_C, as by $\overline{\gamma}_i \subseteq \overline{\gamma}_{i+1} \circ \overline{\gamma}_{i-1} \circ \overline{\gamma}_{i+1}$ the join of any two of them is the natural full order $\leq$ of C.

We have to prove that if $x < y$, and there is a path of core $\overline{\gamma}_i$-edges from x to y, then there is no path of core $\overline{\gamma}_{i+1}$-edges from x to y. Suppose the contrary. Notice that in a path of core edges (of a given $\overline{\gamma}_i$) there is at most one neutral edge, which must precede all the up edges, and must be preceded by all the down edges. If there is no neutral edge, any down edge must precede any up edge.

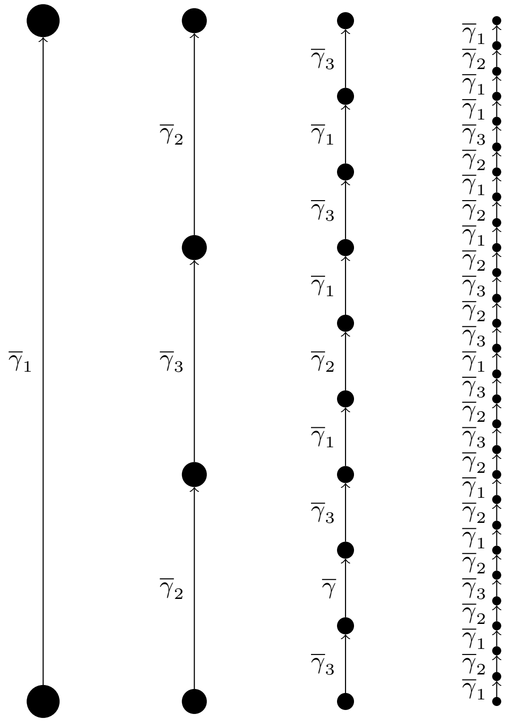

FIGURE 2. The core edges of the $\overline{\gamma}_i$

Suppose that $\frac{a}{3^j} \le x < y \le \frac{a+1}{3^j}$ (where j is minimal).

If there are only up edges in the core $\overline{\gamma}_i$-path from x to y, then the length of any edge of this path is at most the third of the length of the preceding edge. This means that the first edge is $(\frac{3a}{3^{j+1}}, \frac{3a+1}{3^{j+1}})$. Any neutral or down core $\overline{\gamma}_{i+1}$-edge from $\frac{3a}{3^{j+1}}$ is at least $\frac{1}{3^j}$ long, which is longer than the length of the $\overline{\gamma}_i$-path. Hence the $\overline{\gamma}_{i+1}$-path must contain only up edges. This is impossible: the first edge of this path is either at least thrice, or at most third the length of the first edge of the $\overline{\gamma}_i$-path. In the first case the length of the $\overline{\gamma}_{i+1}$-edge is at least twice the length of the $\overline{\gamma}_i$-edge, in the second case at most half of it.

Thus, the path of core $\overline{\gamma}_i$-edges, and likewise, the path of core $\overline{\gamma}_{i+1}$-edges, must start with either a neutral or a down edge. But there is either a single neutral or a single down edge starting at any element of C (besides 1), and that cannot be a core edge of both $\overline{\gamma}_i$ and $\overline{\gamma}_{i+1}$. □

Now, we inject Example 4.1 into a semilattice, namely, FS(C). All elements of FS(C) are intersections of finitely many elements of C, we will refer to those elements as the *factors* of the given semilattice element (so, for example, $0 \wedge 1$ is an element of FS(C) having two factors: 0 and 1).

Note that the $\overline{\gamma}_i$ are binary relations of this semilattice, but they are not quasiorders (not even preorders, as reflexivity fails). In order to get a sublattice in Quo(FS(C)) isomorphic to $\mathbf{M}_3$ we will use a similar construction as in the proof of

Theorem 3.1: from preorders of the free generator, we take the quasiorders generated by them, then add any edge that is in two of them to the third. The difference is that this time, this construction does not yield quasiorders, and so we repeat this step (generating quasiorders, then adding edges so their pairwise intersections coincide) infinitely, then take the limit.

So, denote the quasiorders of $\mathrm{FS}(C)$ generated by $\overline{\gamma}_i$ with $\gamma_i^{(0)}$. Fortunately, these quasiorders are easily understandable (the proof of the following lemma is trivial):

Lemma 4.3. *Let X be a set, $\overline{\delta} \in \mathrm{Quo}(X)$, and $\delta^{(0)}$ the quasiorder of $FS(X)$ generated by $\overline{\delta}$. Then for $c_1, \dots, c_m, d_1, \dots, d_n \in X$, $c_1 \wedge \dots \wedge c_m \xrightarrow{\delta^{(0)}} d_1 \wedge \dots \wedge d_n$ holds iff for each $1 \leq i \leq m$ and each $1 \leq j \leq n$ there are i', j' such that $c_i \xrightarrow{\overline{\delta}} d_{j'}$ and $c_{i'} \xrightarrow{\overline{\delta}} d_j$.* □

Any two of the $\gamma_i^{(0)}$ generate the same quasiorder, but their pairwise intersections do not coincide. This is because by the previous lemma, for any $a, b, c, d \in C$ satisfying

$$a \xrightarrow{\overline{\gamma}_1} b \xleftarrow{\overline{\gamma}_2} c \xrightarrow{\overline{\gamma}_1} d,$$

the edge $(a \wedge c \wedge d, a \wedge b \wedge d)$ is in $\gamma_1^{(0)} \wedge \gamma_2^{(0)}$. It is easy to choose such elements of C:

$$0 \xrightarrow{\overline{\gamma}_1} 1 = \frac{3}{3} \xleftarrow{\overline{\gamma}_2} \frac{2}{3} = \frac{18}{27} \xrightarrow{\overline{\gamma}_1} \frac{19}{27}.$$

As neither 0, 1, or $\frac{19}{27}$ is above $\frac{2}{3}$ in the quasiorder $\overline{\gamma}_3$,

$$(0 \wedge \frac{2}{3} \wedge \frac{19}{27}, 0 \wedge 1 \wedge \frac{19}{27}) \in (\gamma_1^{(0)} \wedge \gamma_2^{(0)}) \setminus \gamma_3^{(0)}.$$

Set recursively for $k > 0$

$$\gamma_i^{(k)} = \gamma_i^{(0)} \vee (\gamma_{i-1}^{(k-1)} \wedge \gamma_{i+1}^{(k-1)}),$$

it is immediate by induction that for each i these form an ascending chain. Let γ_i be their union. The goal is to prove that the γ_i generate a sublattice of $\mathrm{Quo}(\mathrm{FS}(C))$ isomorphic to $\mathbf{M}_3$. The easier part is the following lemma.

Lemma 4.4. *The pairwise meets of the γ_i-s coincide, as do their pairwise joins.*

Proof. For any element $e \in \gamma_{i-1} \wedge \gamma_{i+1}$, there is a k such that $e \in \gamma_{i-1}^{(k)}$ and $e \in \gamma_{i+1}^{(k)}$, hence $e \in \gamma_i^{(k+1)} \subseteq \gamma_i$.

For joins, notice that as the pairwise joins of the $\overline{\gamma}$ coincide, so do the pairwise joins of the $\gamma_i^{(0)}$, denote this join with ν. (Actually, ν is the quasiorder of $\mathrm{FS}(C)$ generated by the full order $<$ on C). Clearly, for any i and any k, $\gamma_i^{(k)} \leq \nu$, thus $\gamma_{i-1} \vee \gamma_{i+1} \leq \nu$. As $\gamma_i \geq \gamma_i^{(0)}$, $\gamma_{i-1} \vee \gamma_{i+1} \geq \nu$ also holds for each i. □

We need yet to show that the γ_i-s do not coincide. The way we obtained the γ_i-s from the $\gamma_i^{(0)}$-s is a little more complicated then the way we obtained the $\gamma_i^{(0)}$ from the $\overline{\gamma}_i$. We need a vocabulary to deal with them, hence the need for the following (atrociously long) definition.

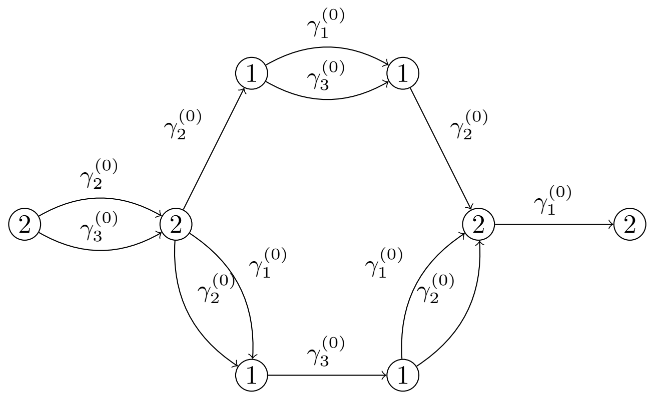

FIGURE 3. Diagram example

Definition 4.5. For a pair $u, v \in \mathrm{FS}(C)$, a nonnegative integer s and $i \in \{1, 2, 3\}$, we say that a pair (K, d), where K is a finite set and d is a mapping $d\colon K \to \{0, 1, \dots, s\}$ (called the *height function*), is a *diagram verifying* $u \xrightarrow{\gamma_i^{(s)}} v$, if the following hold:

- The elements of K are elements of $\mathrm{FS}(C)$ indexed by a nonnegative integer (essentially, K is a subset of $\mathrm{FS}(C)$, with certain elements possibly appearing multiple times),
- there is a natural m such that there are elements $k_0, \dots, k_m \in K$ such that:
  - $k_0 = u$, $k_m = v$, and $k_0, \dots, k_m$ are precisely the elements of K whose height is s,
  - for any $0 \le j < m$, either $k_j \xrightarrow{\gamma_i^{(0)}} k_{j+1}$, or there are subsets $L_+, L_- \subseteq K$ such that the pairs (L_+, d_+) and (L_-, d_-) are diagrams respectively verifying $k_j \xrightarrow{\gamma_{i+1}^{(s-1)}} k_{j+1}$ and $k_j \xrightarrow{\gamma_{i-1}^{(s-1)}} k_{j+1}$, where d_+ and d_- are just the restrictions of d to L_+ and L_- with the exception that $d_+(k_j)$, $d_+(k_{j+1})$, $d_-(k_j)$ and $d_-(k_{j+1})$ equal $s-1$ instead of s (we call such diagrams *subdiagrams*),
- for any proper subset K' of K, the pair $(K', d|_{K'})$ does not satisfy the preceding property.

It is immediate from the definition that two elements of $\mathrm{FS}(C)$ are in $\gamma_i^{(s)}$ iff there is a diagram verifying it. We make the following observations of diagrams.

Lemma 4.6. *For a diagram (K, d) verifying $u \xrightarrow{\gamma_i^{(s)}} v$, there is both a $\gamma_i^{(0)} \cup \gamma_{i+1}^{(0)}$ path and a $\gamma_i^{(0)} \cup \gamma_{i-1}^{(0)}$ path from k_{j_1} to k_{j_2} in K for any $0 \le j_1 < j_2 \le m$ (so specifically also from u to v).*

Proof. We use induction on s. It is enough to prove that there are such paths from k_j to k_{j+1} for all $0 \le j < m$. If $k_j \xrightarrow{\gamma_i^{(0)}} k_{j+1}$, this is immediate. Otherwise, there are subdiagrams of (K, d) verifying both $k_j \xrightarrow{\gamma_{i-1}^{(s-1)}} k_{j+1}$ and $k_j \xrightarrow{\gamma_{i+1}^{(s-1)}} k_{j+1}$, and by using the inductive assumption we gain the needed paths. □

Lemma 4.7. *Suppose $u, v \in \mathrm{FS}(C)$, and (K, d) a diagram verifying $u \xrightarrow{\gamma_i^{(s)}} v$. Take the denominators of all the factors of the elements of K (written so the nominators and denominators are coprime). The largest of these denominators appears as a denominator of one of the factors of u or v.*

Proof. We use induction on s, without fixing i. Define (for all i) $\gamma_i^{(-1)}$ as the equality relation on $\mathrm{FS}(C)$. Note that this is in accordance with $\gamma_i^{(k)} = \gamma_i^{(0)} \vee (\gamma_{i-1}^{(k-1)} \wedge \gamma_{i+1}^{(k-1)})$.

Now the lemma is obvious for $s = -1$. For $s > -1$, an element of K containing a factor with a maximal denominator either equals a k_j for some $0 \leq j \leq m$, or is in a subdiagram verifying $k_{j'} \xrightarrow{\gamma_{i'}^{(s-1)}} k_{j'+1}$ for some $0 \leq j' < m$ and $i' \in \{1, 2, 3\}$. By the inductive assumption, in the latter case there also is a k_j having a factor with a maximal denominator.

Suppose that $k_j = c_1 \wedge \cdots \wedge c_t$, where $c_1 = \frac{b}{3^r}$, with 3^r being the largest denominator appearing among the factors of the elements of K, and b not being divisible by 3. We will assume that the remainder of b modulo 3 is 1 (the other case is similar, only it yields that u has a factor with denominator 3^r). Thus $(\frac{b}{3^r}, \frac{b+1}{3^r})$ is a neutral core edge of $\overline{\gamma}_{i''}$ for an $i'' \in \{1, 2, 3\}$.

By Lemma 4.6, there is both a $\gamma_i^{(0)} \cup \gamma_{i-1}^{(0)}$ path and a $\gamma_i^{(0)} \cup \gamma_{i+1}^{(0)}$ path from k_j to v in K.

Suppose first that i differs from i'', in this case we will consider the first of these paths.

By Lemma 4.3, the path

$$k_j = d_0 \xrightarrow{\gamma_i^{(0)}} d_1 \xrightarrow{\gamma_{i-1}^{(0)}} d_2 \xrightarrow{\gamma_i^{(0)}} \dots \xrightarrow{\gamma_{i-1}^{(0)}} d_h = v$$

in K implies a path

$$\frac{b}{3^r} = e_0 \xrightarrow{\overline{\gamma}_i} e_1 \xrightarrow{\overline{\gamma}_{i-1}} e_2 \xrightarrow{\overline{\gamma}_i} \dots \xrightarrow{\overline{\gamma}_{i-1}} e_h$$

in C, with e_t being a factor of d_t for all $0 \leq t \leq h$.

There is no $\overline{\gamma}_i$ edge, and at most one $\overline{\gamma}_{i-1}$ edge from $\frac{b}{3^r}$ whose target has denominator not greater than 3^r: the edge $(\frac{b}{3^r}, \frac{b+1}{3^r})$. If this is indeed a $\overline{\gamma}_{i-1}$ edge (this is so in the $i'' = i - 1$ case), then $(\frac{b+1}{3^r}, \frac{b+2}{3^r})$ is a (down) $\overline{\gamma}_{i+1}$ edge, which means that there is no $\overline{\gamma}_i$ or $\overline{\gamma}_{i-1}$ edge from $\frac{b+1}{3^r}$ with a target whose denominator is not greater than 3^r. Therefore, all the e_t are equal to $\frac{b}{3^r}$ or $\frac{b+1}{3^r}$ (those equal to the first preceding in the above path those equal to the second).

Now suppose that $i = i''$, and consider the $\gamma_i^{(0)} \cup \gamma_{i+1}^{(0)}$ path. As in the first case, we have the paths

$$k_j = d_0 \xrightarrow{\gamma_i^{(0)}} d_1 \xrightarrow{\gamma_{i+1}^{(0)}} d_2 \xrightarrow{\gamma_i^{(0)}} \dots \xrightarrow{\gamma_{i+1}^{(0)}} d_h = v$$

and
$$\frac{b}{3^r} = e_0 \xrightarrow{\overline{\gamma}_i} e_1 \xrightarrow{\overline{\gamma}_{i+1}} e_2 \xrightarrow{\overline{\gamma}_i} \dots \xrightarrow{\overline{\gamma}_{i+1}} e_h,$$
with e_t being a factor of d_t for all $0 \leq t \leq h$. Now $(\frac{b}{3^r}, \frac{b+1}{3^r})$ is a $\overline{\gamma}_i$ edge, but $(\frac{b+1}{3^r}, \frac{b+2}{3^r})$ is neither a $\overline{\gamma}_i$ or $\overline{\gamma}_{i+1}$ edge, and again all the e_t are equal to either $\frac{b}{3^r}$ or $\frac{b+1}{3^r}$.

As e_h is a factor of v, and (in both cases) equals either $\frac{b}{3^r}$ or $\frac{b+1}{3^r}$ where the remainder of b modulo 3 is 1, the proof is finished. □

Lemma 4.8. $\gamma_1 \neq \gamma_2$.

Proof. Suppose $0 \xrightarrow{\gamma_2} 1$. There must be a diagram verifying it, and by Lemma 4.7 that diagram can only contain elements of $\mathrm{FS}(C)$ that have only factors with denominator 1. Such a diagram can only contain 0, 1, or $0 \wedge 1$, and it is obvious that no such diagram exists. □

With the preceding lemma, we have proved the following theorem.

Theorem 4.9. $\mathrm{Quo}(\mathrm{FS}(\omega))$ *contains a sublattice isomorphic to* $\mathbf{M}_3$. □

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