TYPES FOR QUASIORDERS

We want to generalize the notion of the *type* of a prime quotient in the congruence lattice of a finite algebra into its quasiorder lattice. Note that a prime qutient in the congruence lattice is not necessarily a prime quotient in the quasiorder lattice.

1. Preliminaries

The notion of minimality can be generalized for quasiorders effortlessly, see [QLAT]. We need the following two propositions.

Proposition 1. *Let* $\mathbf{B}$ *be a finite algebra and* $I[\alpha,\beta]$ *an interval in* $\operatorname{Quo}\mathbf{B}$*. If* U *is an* (α,β)*-minimal set such that it is the range of an idempotent unary polynomial* e *of* $\mathbf{A}$ *satisfying* $e(\beta)\not\subseteq\alpha$*, then the mapping from* $\operatorname{Quo}\mathbf{A}$ *to* $\operatorname{Quo}\mathbf{A}|_U$ *that maps each quasiorder of* $\mathbf{A}$ *to its restriction to* U*, is a lattice homomorphism.*

Proposition 2. *If* $\mathbf{B}$ *is a finite algebra,* $\alpha\prec\beta$ *in* $\operatorname{Quo}\mathbf{B}$*, and* U_1 *and* U_2 *are* (α,β)*-minimal sets of* $\mathbf{B}$*, then the algebras* $\mathbf{A}|_{U_1}$ *and* $\mathbf{A}|_{U_2}$ *are polinomially isomorphic, and both are images of an idempotent unary polynomial* e *satisfying* $e(\beta)\not\subseteq\alpha$*.*

2. Algebras minimal to a non-prime congruence quotient

Given a prime quasiorder quotient, we wish to find a congruence quotient corresponding to it, so the former may inherit the type of the latter. In the following section, we will find that corresponding congruence quotient, but it will not generally be a prime quotient. On the other hand, assuming minimality for the quasiorder quotient will essentially also mean minimality for the congruence quotient.

The following definition is, for technical reasons, more liberal than [H-M 4.16].

Definition 3. A *pseudo-meet operation* for an element a of an algebra is any binary polynomial p which satisfies the equations $p(a,x)=p(x,a)=p(x,x)=x$.

For quasiorders $\gamma<\delta$, *a pseudo-meet operation for the quotient* (γ,δ) is a pseudo-meet operation for a sink or a source of a $\delta\backslash\gamma$-edge. A *pseudo-meet–pseudo-join pair* for this qutient is a collection of two pseudo-meet operations, one for the source, one for the sink of a $\delta\backslash\gamma$-edge.

Proposition 4 (HM 4.15., 4.17.)**.** *If* (γ,δ) *is a congruence prime quotient of type 3, 4 or 5, there is a pseudo-meet operation for this quotient. If it is of type 3, or 4, then there is a pseudo-meet–pseudo-join pair for it.*

Proposition 5. *Suppose* $\mathbf{A}$ *is* (α,β)*-minimal, and* $\alpha\not\prec\beta$ *in* $\operatorname{Con}\mathbf{A}$*. Then the interval* $[\alpha,\beta]$ *omits types 3, 4, and 5.*

Proof. Suppose a prime quotient (μ,ν) in the interval is of type 3, 4, or 5. Then there is a pseudo-meet operation p for $\nu\backslash\mu$, meaning an edge $(a,b)\in\nu\backslash\mu\cup(\nu\backslash\mu)^{-1}$ such that all $x\in A$ satisfies $p(a,x)=p(x,a)=p(x,x)=x$.

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Take a pair $(x_1, x_2) \in \beta\backslash\alpha$. We can assume that the element $p(x_1, x_2)$ is not in the α-class of x_1 (if it is, switch x_1 and x_2). Thus the polynomial $p(x_1, x)$ does not map the β-edge (x_1, x_2) into an α-edge, hence it is bijective. As $p(x_1, x_1) = p(x_1, a) = x_1$, this means that $x_1 = a$.

Now, $(b, x_2) \in \alpha$, otherwise the preceding would yield that either $b = a$ or $x_2 = a$. Therefore, only one β-block is not an α-block, and that contains exactly two α-blocks, so obviously $\alpha \prec \beta$. □

Proposition 6. *[H-M 4.20] If (γ, δ) is a congruence prime quotient of type 2, then there is an idempotent ternary polynomial m such that for any x that is in the (γ, δ)-body of the $\mathbf{A}$, and any $y \in A$, $m(x, x, y) = m(y, x, x) = x$. (Consequently, m is a Malcev-operation on the (γ, δ)-body of the $\mathbf{A}$.)*

Lemma 7. *Suppose that $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_n$ in $\operatorname{Con}\mathbf{A}$, $\mathbf{A}$ is (α_0, α_n)-minimal, and all the (α_i, α_{i+1}) types are 2. Then for each pair $(a, b) \in \alpha_n$, there is a bijective unary polynomial of $\mathbf{A}$ mapping a/α_0 to b/α_0.*

Proof. For all $1 \leq i < n$, two distinct α_i-classes in the same α_{i+1}-class can be mapped into one another by a unary polynomial, as there is an addition (modulo α_i) on any (α_i, α_{i+1})-trace of the algebra. By minimality, this polynomial is bijective. Using a sucession of such polynomials, we can map any α_0-class into any other of its α_n-class. □

3. Definition of types for quasiorder quotients

Suppose that $\mathbf{A}$ is a finite algebra, $\alpha \prec \beta$ in $\operatorname{Quo}\mathbf{A}$. All the induced (α, β)-minimal algebras of $\mathbf{A}$ are isomorphic, so it is sufficient to define the type of (α, β) in the case where $\mathbf{A}$ is (α, β)-minimal (otherwise, the algebra will inherit the type of the algebra induced by a minimal set). We will differentiate between two cases.

Definition 8. Suppose $\mathbf{A}$ is (α, β)-minimal, and $\alpha^* \neq \beta^*$. If $\alpha^* \prec \beta^*$ in $\operatorname{Con}\mathbf{A}$, then set $\operatorname{typ}(\alpha, \beta) = \operatorname{typ}(\alpha^*, \beta^*)$. Otherwise, set $\operatorname{typ}(\alpha, \beta) = 1$.

This may seem a little heavy-handed, as there are algebras minimal to one of their non-prime quotients omitting 1 (the most basic example is a multidimensional vector space), but as the next proposition shows, these non-prime qutients cannot be the respective congruence parts of a quasiorder prime quotient.

Proposition 9. *If $\alpha \prec \beta$ in $\operatorname{Quo}\mathbf{A}$, $\mathbf{A}$ is (α, β)-minimal, and there is a congruence γ such that $\alpha^* < \gamma < \beta^*$, then the interval $[\alpha^*, \beta^*]$ does not omit type 1, more precisely, $\operatorname{typ}(\alpha^*, \mu) = 1$ for any congruence $\alpha^* \prec \mu < \beta^*$.*

Proof. From $\mu < \beta$ and $\mu \not\leq \alpha$ we deduce $\alpha \vee \mu = \beta$. This means that there exists an α-*crossedge*, i.e. an α-edge between different μ-classes. $\mathbf{A}$ is of course also (α^*, β^*)-minimal, and so by Lemma 7, this crossedge can be mapped bijectively so its source is mapped into any α^*-class of its β^*-class. Hence, there is an α-crossedge from every α^*-class in that β^*-class, and so there is an α-crossedge from every *element* of that β^*-class. But that means that there is a circle of α-crossedges, and by that, we get that there is an α^*-class intersecting μ-classes, a contradiction. □

Now assume $\alpha^* = \beta^*$. First we need to understand what (α, β)-minimality means in this case. Take an edge (x, y) of $\beta\backslash\alpha$ such that $x/\beta^* \prec y/\beta^*$ in β.

Proposition 10. *If (a,b) is a $\beta\backslash\alpha$-edge, then $a/\beta^* \prec_{\beta/\beta^*} b/\beta^*$. Furthermore, if (c,d) is an other $\beta\backslash\alpha$-edge, then $(a/\beta^*, b/\beta^*)$ and $(c/\beta^*, d/\beta^*)$ can be mapped into each other by an automorphism of the poset A/β^*.*

Proof. Obviously, $a/\beta^* \neq b/\beta^*$. There are elements u' and v' in $\mathbf{A}/\beta^*$ such that $a \leq u' \prec v' \leq b$ in β/β^*, and $(u',v') \notin \alpha/\beta^*$. Take any representants u and v of u' and v' respectively. As $\alpha \prec \beta$ in $\operatorname{Quo}\mathbf{A}$, there are elements $u = c_0, \dots, c_l = v$ in A such that each (c_i, c_{i+1}) is in α, or is a polynomial image of the edge (a,b). There edges cannot all be in α, and by (α,β)-minimality the one that is not in α is the image of (a,b) by a bijective polynomial. The inverse of this polynomial maps (u',v') into $(a/\beta^*, b/\beta^*)$, which finishes the proof (bijective polynomials are automorphisms of compatible relations).

The second statement is a result of the same argument: simply consider that β is also generated by $\alpha \cup \{(c,d)\}$. □

Definition 11. The *enlargement of* $\mathbf{A}$ *by the quasiorder* β (denoted by $\mathbf{A}_+$) is the subalgebra of $\mathbf{A}^3$ consisting of triples (a,b,c) satisfying $(a,b),(b,c) \in \beta$. For an arbitrary $\delta \in \operatorname{Quo}\mathbf{A}$, the enlargement of δ is a congruence of $\mathbf{A}_+$ defined by

$$\delta_+ = \operatorname{Tr}(\{((a,b,c),(a,b',c)) \in \mathbf{A}_+^2 : (b,b') \in \delta \cup \delta^{-1}\}).$$

Informally, two triples are in δ_+ if their first and last components coincide, and there is a $\delta \cup \delta^{-1}$-path in A between their middle components that is entirely in between the shared first and last component in β. It is easy to see that δ_+ is indeed a congruence of $\mathbf{A}_+$.

Proposition 12. *The mapping $\delta \mapsto \delta_+$*

- *is a $\vee$-endomorphism from* $\operatorname{Quo}\mathbf{A}$ *into* $\operatorname{Con}\mathbf{A}_+$*,*
- *maps a quasiorder $\delta \geq \beta$ into the product congruence $0_A \times 1_A \times 0_A$ (so $(a_1,a_2,a_3)\delta_+(b_1,b_2,b_3)$ iff $a_1 = b_1$ and $a_3 = b_3$),*
- *maps α and β into different congruences, moreover,*

$$\beta_+\backslash\alpha_+ = \{((c,a,d),(c,b,d)) : (a,b) \in (\beta\backslash\alpha) \cup (\beta\backslash\alpha)^{-1},\ c\beta^* a,\ d\beta^* b\}.$$

Proof. The first two points are easy deductions from the definition. We will only prove the formula of the third point.

Any element of the right hand side is immediately in β_+. That they are not in α_+ follows from the fact that (by Proposition 10) $c/\beta^* \prec_{\beta/\beta^*} d/\beta^*$, so the elements of A between c and d in β are the elements of the β^*-blocks of c and d. Among these elements, there cannot be an $\alpha \cup \alpha^{-1}$-path between a and b, because that would mean that $(a,b) \in \alpha$.

Conversely, take an element $((k,x,l),(k,y,l))$ of $\beta_+\backslash\alpha_+$. Note that $((k,x,l),(k,l,l))$ and $((k,l,l),(k,y,l))$ are both in β_+. One of two (assume that the first) must not be in α_+. This means that $(x,l) \in \beta\backslash\alpha$. Similarly, either (k,x) or (k,y) is in $\beta\backslash\alpha$. The former is impossible by the second statement of Proposition 10, as $(k/\beta^*, x/\beta^*)$ obviously cannot be mapped into $(x/\beta^*, l/\beta^*)$ by an automorphism of A/β^*.

This leaves $(k,y) \in \beta\backslash\alpha$, and, again by Proposition 10, there k/β^* can be mapped into x/β^* by an A/β^*-automorphism. But as (k,x) is in β, we deduce that k and x are in the same β^*-block. Consequently, $(x,y) \in \beta\backslash\alpha$. By the same argument, l and y are in the same β^*-block.

□

Definition 13. Suppose $\mathbf{A}$ is (α, β)-minimal, and $\alpha^* = \beta^*$. If the interval $[\alpha_+, \beta_+]$ in $\operatorname{Con} \mathbf{A}_+$ does not omit 4, then we set $\operatorname{typ}(\alpha, \beta) = 5$, otherwise, $\operatorname{typ}(\alpha, \beta) = 5$ if the interval does not omit 5, and $\operatorname{typ}(\alpha, \beta) = 1$ if it does.

Admittedly, this definition is premature: we will soon prove that the interval $[\alpha_+, \beta_+]$ necessarily omits 2 and 3, and cannot contain both types 4 and 5. It is convenient, though, to be able to speak of quasiorder types even before this is proved.

4. Properties of different types

By the definition of the previous section, the type of a prime quasiorder quotient is inherited from the type of its congruence part, or the type of the congruence parts of the enlargements of the quasiorders. We will refer to a prime quasiorder quotient as *-quotient or +-qutient depending on whether the former or the latter is the case.

First we prove that a prime quotient being a +-quotient is equivalent to the congruence parts of the two quasiorders coinciding (this is true by definition only if the algebra is minimal to this quotient).

Proposition 14. *A prime quotient (α, β) is a +-quotient iff $\alpha^* = \beta^*$.*

Proof. We have to show that if the congruence parts of α and β differ on $\mathbf{A}$, then they also differ on an (α, β)-minimal set M of $\mathbf{A}$. This follows from the fact (Proposition 1) that the restriction to M is a lattice homomorphism from $\operatorname{Quo} \mathbf{A}$ to $\operatorname{Quo} \mathbf{A}|_M$, thus $\alpha|_M \vee \beta^*|_M = (\alpha \vee \beta^*)|_M = \beta_M$, so (by $\alpha_M \neq \beta_M$) α^*_M and β^*_M must differ. □

To understand types in the +-case, first we need the following lemma. It informally states that when we "descend" to a minimal set of an enlargement, we do not lose too much information, and the minimal set is at least as large as the original (minimal) algebra.

Lemma 15. *If $\mathbf{A}$ is (α, β)-minimal, then for any (α_+, β_+)-minimal set M of $\mathbf{A}_+$ any $c \in A$, c is a middle component of one of the elements of M.*

Proof. Take any polynomial $p \in \operatorname{Pol}_1 \mathbf{A}_+$ that does not map β_+ into α_+. The middle component of p is itself a unary polynomial of $\mathbf{A}$, so by the (α, β)-minimality of $\mathbf{A}$, it is enough to show that the middle component does not map β into α. We do that by proving that if $((a, b, c), (a, b', c)) \in \beta_+ \backslash \alpha_+$, then $(b, b') \in \beta \cup \beta^{-1}$.

Assume for the sake of simplicity that $\beta^* = 0_{\mathbf{A}}$ (otherwise, take each element's β^*-class). Notice that by the (α, β)-minimality of $\mathbf{A}$, and the fact that $\alpha \prec \beta$, one concludes that the $\beta \backslash \alpha$ edges can be mapped into each other by bijective polynomials. Also, these edges are covering edges of β (because otherwise all the covering edges, being not bijective images of this edge, would be in α, but then α would have to coincide with β).

Suppose that $((a, b, c), (a, b', c)) \in \beta_+ \backslash \alpha_+$. Either (a, b) or (a, b') must be in $\beta \backslash \alpha$, and also either (b, c) or (b', c), and this can happen only if (a, c) is a covering edge in β, and $\{a, c\} = \{b, b'\}$. □

Now we can state the following crucial property of +-quotients.

Lemma 16. *Suppose* $\mathbf{A}$ *is* (α,β)*-minimal, where* (α,β) *is a +-quotient. Then for any* $\alpha_+ \leq \mu \prec \nu \leq \beta_+$ *(in* $\operatorname{Con}\mathbf{A}_+$*), the type of the congruence quotient* (μ,ν) *cannot be 2 or 3. If the type is 5, there is a pseudo-meet operation, if 4, a pseudo-meet–pseudo-join pair for the quasiorder quotient* (α,β)*.*

Proof. If the type was 2 or 3, there would be a polynomial of $\mathbf{A}_+$ mapping a $\beta_+\backslash\alpha_+$ edge into a $(\beta_+\backslash\alpha_+)^{-1}$ edge. By Proposition 12, the middle component of this polynomial would map a $\beta\backslash\alpha$ edge into a $(\beta\backslash\alpha)^{-1}$ edge (in the case of type 2, note that a ν-block cannot contain more than two μ-blocks).

The statement for types 4 and 5 follows from the properties of the pseudo-meet and pseudo-join polynomials for the quotient (μ,ν) in the algebra $\mathbf{A}_+|_M$, Lemma 15 and the fact that M must contain an (α_+,β_+)-minimal set. □

Corollary 17. *If a prime quasiorder quotient is of type 2 or 3, then it is a *-quotient.*

As a pseudo-meet operation for (γ,δ) is also one for (γ^*,δ^*):

Corollary 18. *If an algebra is minimal with respect to a prime quasiorder quotient of type 3,4 (of type 5), then there is a pseudo-meet–pseudo-join pair (a pseudo-meet operation) for it.*

Corollary 19 (THIS IS KNOWN ALREADY BY ?????)**.** *If a locally finite variety omits types 1,4, and 5, then it does not have non-congruence quasiorders, hence it is congruence* n*-permutable for some* n*.*

Lemma 20. *If* $\mathbf{A}$ *is minimal to the type 4 (type 5) +-quotient* (α,β)*, then* $\beta\backslash\alpha$ *contains only a single edge (contains edges with either a common source or a common sink).*

Proof. Suppose that $(a_0,b_0)\in\beta\backslash\alpha$ such that there is a pseudo-meet operation p for a_0. Take an arbitrary $(a,b)\in\beta\backslash\alpha$, and notice that either $(a,p(a,b))$ or $(p(a,b),b)$ is also in $\beta\backslash\alpha$. In the first case, the unary polynomial $p(a,x)$ must be bijective by (α,β)-minimality of $\mathbf{A}$, and as $p(a,a_0)=p(a,a)=a$, a must be equal to a_0. In the second case, the same argument shows that $b=a_0$, which is impossible by Proposition 10. So all the $\beta\backslash\alpha$-edges have a shared source. If there is a pseudo-meet operation for b_0 instead, then all the $\beta\backslash\alpha$-edges will have a shared sink. □

So, as we promised, the interval $[\alpha_+,\beta_+]$ cannot contain both types 4 and 5.

Corollary 21. *If* $\mathbf{A}$ *is minimal to the type 4 +-quotient* (α,β)*, then* $\alpha_+\prec\beta_+$ *in* $\operatorname{Con}\mathbf{A}_+$*.*

Proof. By Lemma 20, there is only one $\beta\backslash\alpha$-edge, obviously, both its source and sink must be a singleton β^*-class. By Proposition 12, $\beta_+\backslash\alpha_+$ is a single double edge. □

Lemma 22. *If* $\mathbf{A}$ *is minimal to a quasiorder quotient* (α,β)*, and there is a pseudo-meet operation* p *for* $a\in A$*, then all* $\beta\backslash\alpha$*-edges have* a *as either source or sink.*

Proof. For any $a\neq x_0\in A$, the unary polynomials $p(x,x_0)$ and $p(x_0,x)$ are not bijective (mapping both x_0 and a into x_0), therefore they map δ into γ. This means that for any $(d_1,d_2)\in\delta\backslash\gamma$, either $d_1=a$ or $d_2=a$, because otherwise, $d_1\xrightarrow{\gamma}p(d_1,d_2)\xrightarrow{\gamma}d_2$. □

Lemma 16 has a converse.

Lemma 23. *If* $\mathbf{A}$ *is minimal to the* $+$*-quotient* (α, β)*, and there is a pseudo-meet operation for this quotient, then* $\operatorname{typ}(\alpha, \beta) \in \{4, 5\}$*. If there is a psudo-meet–pseudo-join pair for this quotient, then* $\operatorname{typ}(\alpha, \beta) = 4$*.*

Proof. We only prove the first statement of the lemma, as the second is much the same.

Take a pair of congruences $\alpha_+ \leq \mu \prec \nu \leq \beta_+$ of $\mathbf{A}_+$, and let M be a (μ, ν)-minimal set. Choose an element of $\beta_+ \backslash \alpha_+ \cap M^2$, by Proposition 12 it has the form $((c, a, d), (c, b, d))$, where $(a, b) \in \beta \backslash \alpha \cup (\beta \backslash \alpha)^{-1}$, and $(a, c), (b, d) \in \alpha^*$. By Lemma 22, the pseudo-meet operation p is for either a or b. We may assume that $(a, b) \in \beta \backslash \alpha$, and that $p(a, x) = p(x, a) = p(x, x) = x$ for all x.

There is an idempotent unary polynomial $e \in \operatorname{Pol}_1 \mathbf{A}_+$ such that $e(A_+) = M$ (REFERENCE). We define the binary polynomial p' on $\mathbf{A}_+|_M$ with

$$p'((x_1, x_2, x_3), (y_1, y_2, y_3)) := e((p(x_1, y_1), p(x_2, y_2), p(x_3, y_3))).$$

As p is idempotent and e is idempotent (in different senses), p' is also idempotent. Furthermore, $p'((c, a, d), (c, b, d)) = p'((c, b, d), (c, a, d)) = (c, b, d)$, so p' is a proper binary polynomial on an (α_+, β_+)-trace of M. This means that $\operatorname{typ}(\alpha, \beta) \neq 1$. As the type of a $+$-quotient cannot be 2 or 3 (Lemma 16), the proof is done. □

We summarise this section in the following theorem:

Theorem 24. *If* $\mathbf{A}$ *is minimal to the prime quotient* (α, β)*, then the type of* (α, β) *is*

- *3, iff* $\beta \backslash \alpha$ *is a single double edge, and there is a pseudo-meet–pseudo-join pair for it (this case is only possible if* $\alpha^* = \beta^*$*) ,*
- *4, iff* $\beta \backslash \alpha$ *is a single (directed) edge, and there is a pseudo-meet–pseudo-join pair for it,*
- *5, iff there is a pseudo-meet operation for it, but not a pseudo-meet–pseudo-join pair (and in this case, the pseudo-meet operation is for either the shared sink or the shared source of all the* $\beta \backslash \alpha$*-edges),*
- *2, iff* (α^*, β^*) *is a prime congruence quotient of type 2,*
- *1 in any other case.*

Proof. The $+$-quotient case is covered by Lemma 16, Lemma 23, and Corollary 17. In the *-case, the statement for type 2 is the first statement of Proposition 9. Finally, the statements for types 3,4, and 5 for *-quotients can be easily deduced by Lemma 22. □

5. Types in a quasiorder lattice

There are two basic conditions about the labeling of congruence lattices: the first is that prime projective quotients must have the same type, the second that the solvability and strong solvability relations must be congruences. The first is also true for the labelings of quasiorder lattices.

Theorem 25. *Suppose that* (α, β) *and* (γ, δ) *are prime projective quotients of* $\operatorname{Quo} \mathbf{A}$*. Then* $\operatorname{typ}(\alpha, \beta) = \operatorname{typ}(\gamma, \delta)$*.*

Proof. As the minimal (α,β) and (γ,δ) sets of the algebra coincide (REFERENCE, MAYBE), we may assume that $\mathbf{A}$ is both (α,β)- and (γ,δ)-minimal. Note that if (α,β) is a $+$-quotient, then so is (γ,δ).

CASE 1: Both are $+$-quotients

In this case, both types are among 1, 4, and 5.

If $\operatorname{typ}(\alpha,\beta)=4$, then by Lemma 20, $|\beta\backslash\alpha|=1$. As $\emptyset\neq\delta\backslash\gamma\subseteq\beta\backslash\alpha$, this means that $\delta\backslash\gamma=\beta\backslash\alpha$, hence there is a pseudo-meet–pseudo-join pair for (γ,δ), so $\operatorname{typ}(\gamma,\delta)=4$.

If $\operatorname{typ}(\alpha,\beta)=5$, then, as there is a pseudo-meet operation for *any* $\beta\backslash\alpha$-edge, there is also one for (γ,δ). $\operatorname{typ}(\gamma,\delta)=4$ is not possible, as it would mean that there is a pseudo-meet–pseudo-join pair for (γ,δ), and it would also be one to (α,β). Therefore $\operatorname{typ}(\gamma,\delta)=5$.

Finally, if $\operatorname{typ}(\alpha,\beta)=1$, there must not exist a pseudo-meet operation for (γ,δ), because it would also be one for (α,β), so $\operatorname{typ}(\gamma,\delta)=1$.

CASE 2: Both are *-quotients

First, assume that neither of the two types is 1. Then $\alpha^*\prec\beta^*$, and $\gamma^*\prec\delta^*$ (in the congruence lattice). Notice that $\alpha^*\wedge\delta^*=\gamma^*$ (as $\delta\mapsto\delta^*$ is a $\wedge$-homomorphism) and $\alpha^*\vee\gamma^*=\beta^*$ (if $\alpha^*\vee\gamma^*$ was α^*, then $\alpha^*\wedge\delta^*$ would be γ^*). Therefore, $\operatorname{typ}(\alpha,\beta)=\operatorname{typ}(\alpha^*,\beta^*)=\operatorname{typ}(\gamma^*,\delta^*)=\operatorname{typ}(\gamma,\delta)$.

Now assume that $1=\operatorname{typ}(\gamma,\delta)\neq\operatorname{typ}(\alpha,\beta)$. Again, $\alpha^*\wedge\delta^*=\gamma^*$, and $\alpha^*\vee\gamma^*=\beta^*$, because (α^*,β^*) is a prime congruence quotient. Choose a congruence ρ such that $\gamma^*\prec\rho\leq\delta^*$. By Proposition 9, $\operatorname{typ}(\gamma^*,\rho)=1$. But by prime projectivety, $\operatorname{typ}(\alpha^*,\beta^*)=\operatorname{typ}(\gamma^*,\rho)$, a contradiction.

Finally, assume that $\operatorname{typ}(\gamma,\delta)\neq\operatorname{typ}(\alpha,\beta)=1$. Choose a congruence τ such that $\alpha^*\leq\tau\prec\alpha^*\vee\delta^*$. The type of $(\tau,\alpha^*\vee\delta^*)$ cannot be 3, 4, or 5, because that (along with the (α^*,β^*)-minimality of the algebra) would mean by Proposition 5 that $\tau=\alpha^*$, $\alpha^*\vee\gamma^*=\beta^*$, and so $\operatorname{typ}(\alpha^*,\beta^*)\in\{3,4,5\}$. Hence, $\operatorname{typ}(\gamma,\delta)=\operatorname{typ}(\gamma^*,\delta^*)=\operatorname{typ}(\tau,\alpha^*\vee\delta^*)=2$, because (γ^*,δ^*) and $(\tau,\alpha^*\vee\delta^*)$ are prime projective.

Now choose a congruence ρ such that $\alpha^*\prec\rho\leq\alpha^*\vee\delta^*$. By Proposition 9, $\operatorname{typ}(\alpha^*,\rho)=1$. Consider an (α^*,ρ)-trace D. Denote with C the $\alpha^*\vee\delta^*$-class that contains this D.

Notice that any α^*-class of C must contain an element of the (γ^*,δ^*)-body of the algebra (otherwise, C would be an α^*-class). Consider the pseudo-Malcev operation d for the congruence quotient (γ^*,δ^*). As any α^*-class of D contains an element that is in the (γ^*,δ^*)-body, any α^*-class of D, as well as D itself, is a subalgebra with respect to the polynomial d (because d is idempotent on the (γ^*,δ^*)-body). Also, d acts as a Malcev-operation on the α^*-classes of D: if A_1 and A_2 are two such classes, then there are $a_1\in A_1$ and $a_2\in A_2$ such that both a_1 and a_2 is in the (γ^*,δ^*)-body, thus $d(a_1,a_1,a_2)=d(a_2,a_1,a_1)=a_2$, and any element of $d(A_1,A_1,A_2)$ or $d(A_2,A_1,A_1)$ must be in the same α^*-class as a_2. This contradicts $\operatorname{typ}(\alpha^*,\rho)=1$.

CASE 3: (α,β) is a *-quotient, while (γ,δ) is a $+$-quotient

If $\operatorname{typ}(\gamma,\delta)=4$, $\delta\backslash\gamma$ is a single edge (a,b) by Lemma 24. As $\alpha^*\neq\beta^*$, there is an edge $(c,d)\in\beta^*\backslash\alpha$. There is an α-γ path from c to d, which is possible in only one fashion: if

$$c\xrightarrow{\alpha}a\xrightarrow{\delta}b\xrightarrow{\alpha}d.$$

Consider the pseudo-meet operation p for a (so $p(a,x) = p(x,a) = p(x,x) = x$ for all x), and the path

$$c = p(c,a) \xrightarrow{\delta} p(c,b) \xrightarrow{\alpha} p(a,d) = d,$$

to deduce that $c = a$. Likewise, from the pseudo-meet operation for b we can deduce that $d = b$. So the pseudo-meet–pseudo-join pair for (γ, δ) acts on the $(\alpha^*, \beta*)$-trace as lattice operations, thus $\mathrm{typ}(\alpha, \beta) = 3$ (the type is not 3, because that would mean that (d,c) is also in $\beta^* \backslash \alpha$, which would yield that (d,c) is also coinciding with (a,b)).

If $\mathrm{typ}(\gamma, \delta) = 5$, we can assume that the edges in $\delta \backslash \alpha$ share a common source a, and there is a pseudo-meet operation p for it (again by Lemma 24). As in the previous case, we deduce that for an arbitrary $(c,d) \in \beta^* \backslash \alpha$, c is coinciding with a, furthermore, there is a $b \in A$ such that

$$a \xrightarrow{\delta} b \xrightarrow{\alpha} d.$$

This means that there is a pseudo-meet operation for (c,d), we need only to prove that it has no pseudo-join pair.

Assume contrariwise that q is a bijective polynomial such that $q(d,x) = q(x,d) = q(x,x) = x$. Now consider

$$a = q(a,a) \xrightarrow{\delta} q(a,b) \xrightarrow{\alpha} q(a,d) = a,$$

to see that $q(a,b)$ is in the β^*-class of a, but that class only contains a and d (as the assumption was $\mathrm{typ}(\alpha^*, \beta^*) = 4$). If $q(a,b) = d$, then $(a,d) \in \delta \backslash \gamma$, which is a contradiction, as there is a pseudo-meet–pseudo-join pair for (a,d). So $q(a,b) = a$, and likewise, $q(b,a) = a$, thus there is a pseudo-meet–pseudo-join pair for (a,b), again a contradiction.

Finally, set $\mathrm{typ}(\gamma, \delta) = 1$. If $\mathrm{typ}(\alpha, \beta) \in \{3,4,5\}$, then we can assume there is an edge $(a,b) \in \beta^* \backslash \alpha$ and a pseudo-meet operation p for a. By Proposition 22, taking an arbitrary $\delta \backslash \gamma$-edge, a is either a source or a sink of it, so there is a pseudo-meet operation for (γ, δ), which contradicts $\mathrm{typ}(\gamma, \delta) = 1$ by Lemma 24.

If $\mathrm{typ}(\alpha, \beta) = 2$, there is a pseudo-Malcev operation m for (α^*, β^*). Take an arbitrary edge $(a,b) \in \beta^* \backslash \alpha^*$, there is an α-$\delta \backslash \gamma$ path from a to b:

$$a \xrightarrow{\alpha} c_1 \xrightarrow{\delta \backslash \gamma} \dots \xrightarrow{\alpha} c_k \xrightarrow{\delta \backslash \gamma} b$$

This path is entirely in β, and as it is between elements of the same β^*-class, it must lie entirely in that β^*-class. Therefore, c_1 and c_2 are in the (α^*, β^*)-body, so $(c_2, c_1) = (m(c_1, c_1, c_2), m(c_1, c_2, c_2)) \in \delta$, and $(c_1, c_2) \in \delta^* \backslash \gamma^*$, which contradicts the fact that (γ, δ) is a $+$-quotient. $\square$

On the other hand, the concept of solvability does not seem to extend to quasiorder lattices. Consider the semigroup $\mathbf{S}$ with the following multiplication table:

	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	1	2	2
3	0	1	2	3

The usual definition of solvability (α and β being in the same block iff $\operatorname{typ}[\alpha \wedge \beta, \alpha \vee \beta] \subseteq \{1, 2\}$) does not yield a congruence in this case. Neither does the definition of strong solvability (α and β being in the same block iff $\operatorname{typ}[\alpha\wedge\beta, \alpha\vee\beta] \subseteq \{1\}$).

Proposition 26. *There are* $\alpha, \beta, \gamma, \delta \in \operatorname{Quo}\mathbf{S}$ *such that* $\alpha \prec \beta$, $\gamma \prec \delta < \beta \vee \gamma$, $\operatorname{typ}(\alpha, \beta) = 1$*, and* $\operatorname{typ}(\gamma, \delta) = 5$.

Proof. Put

$$\alpha = 0_{\mathbf{S}},\ \overline{\beta} = \{1, 0)\},\ \overline{\gamma} = \{(0, 1), (0, 2), (0, 3), (2, 1)\},$$
$$\overline{\delta} = \{(0, 1), (0, 2), (0, 3), (2, 1), (2, 3)\},$$

with $\overline{\omega}$ denoting the set of non-loop edges of a relation ω.

As the set $\{0, 1\}$ is (α, β)-minimal, and the algebra induced on it is trivial, $\operatorname{typ}(\alpha, \beta) = 1$. $\mathbf{S}$ is itself a (γ, δ)-minimal algebra $(2, 3) \in \delta\backslash\gamma$, and the multiplication is a pseudo-meet operation for 3 (with there being no pseudo-meet operation for 2), $\operatorname{typ}(\gamma, \delta) = 5$ by Theorem 24. □

One may try to alter the definition of the solvability relation for quasiorders (or even the definition of types for quasiorders) to circumvent this problem. The next proposition shows a limit to this approach.

Proposition 27. *There is no congruence* ρ *of* $\operatorname{Quo}\mathbf{S}$ *such that* $\rho|_{\operatorname{Con}\mathbf{S}} = \sim_s$*, more precisely, the congruence of* $\operatorname{Quo}\mathbf{S}$ *generated by the congruence solvability relation is* $1_{\operatorname{Quo}\mathbf{S}}$.

Proof. Let $\sim$ be the congruence of $\operatorname{Quo}\mathbf{S}$ generated by the congruence solvability relation. We will need the following elements of $\operatorname{Quo}\mathbf{S}$:

$$\overline{\tau} = \{(0, 1), (1, 0)\}$$
$$\overline{\eta_1} = \{(0, 1), (0, 2), (2, 1), (3, 1)\}$$
$$\overline{\eta_2} = \{(0, 1), (0, 2), (2, 1), (3, 1), (3, 2)\}$$
$$\overline{\eta_3} = \{(0, 1), (0, 2), (0, 3), (2, 1)\}$$
$$\overline{\eta_4} = \{(0, 1), (0, 2), (0, 3), (2, 1), (2, 3)\}$$
$$\overline{\eta_5} = \{(0, 1), (3, 2)\}$$
$$\overline{\eta_6} = \{(0, 1), (2, 3)\}$$
$$\overline{\eta_7} = \{(0, 1), (0, 2), (2, 1)\}$$

Of course, τ is a congruence such that $(0_{\mathbf{S}}, \tau)$ is a congruence prime quotient of type 1. Therefore, $0_{\mathbf{S}} \sim \tau$, so as $\eta_1 \vee \tau = \eta_2 \vee \tau$ and $\eta_3 \vee \tau = \eta_4 \vee \tau$, $\eta_1 \sim \eta_2$ and $\eta_3 \sim \eta_4$. Note that $\eta_5 \leq \eta_2$, $\eta_6 \leq \eta_4$, and $\eta_1 \wedge \eta_5 = \eta_3 \wedge \eta_6 = \eta_5 \wedge \eta_6$. From these we can deduce that $\eta_5 \sim \eta_5 \wedge \eta_6 \sim \eta_6$.

As $\eta_7 \geq \eta_5 \wedge \eta_6$, we get that

$$\eta_7 = \eta_7 \vee (\eta_5 \wedge \eta_6) \sim \eta_7 \vee \eta_5 \vee \eta_6 \sim \eta_7 \vee \eta_6 \vee \eta_5 \vee \tau = 1_{\mathbf{S}}.$$

But as $\sim_s$ is closed to inversion (meaning that $\delta_1 \sim_s \delta_2$ implies $\delta_1^{-1} \sim_s \delta_2^{-1}$), $\sim$ must also be closed to inversion. Thus, $\eta_7^{-1} \sim 1_{\mathbf{S}}$, and $0_{\mathbf{S}} = \eta_7 \wedge \eta_7^{-1} \sim 1_{\mathbf{S}}$, so $\sim$ is the full relation on $\operatorname{Quo}\mathbf{S}$. □

We note that the above proof did not use the simmetry of $\sim$, so the quasiorder of $\operatorname{Quo}\mathbf{S}$ generated by the congruence solvability relation is also the full relation.

Problem 28. Is the solvability relation on the quasiorder lattices of algebras that omit 1 for quasiorders/for congruences a lattice congruence? What about varieties omitting 1?

An other difference for quasiorders is that Proposition 5 is not true for them: a counterexample is the two-element lattice, which is obviously a minimal algebra, yet its quasiorder lattice is the direct square of the two-element lattice. A more elaborate counterexample: consider the semilattice $(\{0,1,2,3\},\max)$. If $\overline{\alpha}=\{(0,1),(0,2),(1,2)\}$, and β is the full order $<$, then the semilattice is (α,β)-minimal, but the reader can easily chech that the interval $[\alpha,\beta]$ is the four-element chain. However, there are (in a way) no more counterexamples.

Theorem 29. *Suppose* $\mathbf{A}$ *is* (α,β)*-minimal, and* $\alpha \not\prec \beta$ *in* $\operatorname{Quo}\mathbf{A}$*. Then the interval* $[\alpha,\beta]$ *either:*

- *contains only types 4, and is isomorphic to the direct square of the two-element lattice, or*
- *contains only types 5, and is a distributive lattice, or*
- *contains only types 1 and 2.*

Proof. First suppose that (γ_1,γ_2) is a type 3 or a type 4 quasiorder prime quotient in the interval, and take an edge $(a,b)\in\gamma_2\backslash\gamma_1$. By Theorem 24, there are pseudo-meet operations for both a and b, so by Lemma 22, all $\beta\backslash\alpha$ edges have a and b as either source ot sink. Therefore, $\beta\backslash\alpha\subseteq\{(a,b),(b,a)\}$, and by $\alpha\not\prec\beta$ there must be equality. Furthermore, either $\alpha\cup\{(a,b)\}$ or $\alpha\cup\{(b,a)\}$ must be a quasiorder.

We need yet to show that if either is a quasiorder, then both are. So suppose that $\delta_1:=\alpha\cup\{(a,b)\}$ is a quasiorder, and $\delta_2:=\alpha\cup\{(b,a)\}$ is not. Notice that δ_2 is a preorder, as it is in the quasiorder β, and its transitive closure does not contain (a,b), because there is no path from a to b containing only α-edges the edge (b,a). This means that there is a unary polynomial p such that $p(\delta_2)\notin\delta_1$. Any unary polynomial maps all α edges into δ_1 edges, and (b,a) into a β edge, which means that p maps the edge (b,a) into (a,b), but then it also maps (a,b) into (b,a), a contradiction.

So in this case, the interval $[\alpha,\beta]$ is isomorphic to the lattice $\mathbf{2}^2$. By Theorem 24 and only contains types 4, because the quasiorders of any prime quotient in it differ only in a single edge ($(a.b)$ or (b,a)), and there is a pseudo-meet–pseudo-join pair for that edge.

Now suppose that (γ_1,γ_2) is a type 5 quasiorder prime quotient in the interval $[\alpha,\beta]$, and $(a,b)\in\gamma_2\backslash\gamma_1$. We can assume by Theorem 24 that there is a pseudo-meet operation for a, in which case by Lemma 22, all $\beta\backslash\alpha$ edges have a as either source or sink. So there are elements of A $x_1,\dots,x_k,y_1,\dots,y_l$ such that

$$\beta\backslash\alpha=\{(x_1,a),\dots,(x_k,a),(a,y_1),\dots,(a,y_l)\},$$

with k or l possibly being zero.

Suppose there is an i so that there is a pseudo-meet operation for x_i (or y_i). This means that, by Lemma 22, all $\beta\backslash\alpha$ edges have x_i as either source or sink, so $k=1$, and either $l=0$ or $l=1$ and $y_1=x_1$. The first case is impossible by $\alpha\not\prec\beta$, in the second, the quasiorders in any prime quotient in the interval $[\alpha, \beta]$ differ only in one edge, and there is a pseudo-meet–pseudo-join pair for it, which by Theorem 24 contradicts the assumption that this interval contains a type 5.

Therefore, there is no pseudo-meet operation for any of the x_i or the y_i. By Theorem 24, this means that the interval $[\alpha, \beta]$ contains only types 5. To prove that this interval is a distributive lattice, it is enough to note that for quasiorders $\alpha \leq \delta_1, \delta_2 \leq \beta$, $\delta_1 \vee \delta_2 = \delta_1 \cup \delta_2$, because for any $1 \leq i \leq k$ and $1 \leq j \leq l$, the edge (x_i, y_j) is in α (as it is in β and a is neither its sink or source). □

About the third case, we can say a little more, if the interval omits type 1, namely, that in that case the interval is a modular lattice. This is true for congruences without assuming minimality:

Theorem 30. *[H-M 6.8] Suppose that $\alpha < \beta$ in $\operatorname{Con} \mathbf{A}$, and all the congruence types in the interval $N := [\alpha, \beta]$ are 2. Then N is a modular lattice.*

Theorem 31. *Suppose that $\alpha < \beta$ in $\operatorname{Quo} \mathbf{A}$, $\mathbf{A}$ is (α, β)-minimal, and all the quasiorder types in the interval $N := [\alpha, \beta]$ are 2. Then N is a modular lattice.*

Proof. Consider the mapping $\delta \mapsto \delta^*$ from N into $\operatorname{Con} \mathbf{A}$. This is a $\wedge$-homomorphism, and preserves the $\prec$ relation by Proposition 9. This means that it is also injective.

Claim 1. *It is also a $\vee$-homomorphism.*

Suppose $\delta_1, \delta_2 \in N$. Then

$$\delta_1^* \vee \delta_2^* \vee (\delta_1 \wedge \delta_2) = \delta_1 \wedge \delta_2.$$

To see this, notice that the left side is in N, and

$$(\delta_1^* \vee \delta_2^* \vee (\delta_1 \wedge \delta_2))^* \geq \delta_1^*,$$

thus, $\delta \mapsto \delta^*$ being a $\wedge$-homomorphism and so order-preserving,

$$\delta_1^* \vee \delta_2^* \vee (\delta_1 \wedge \delta_2) \geq \delta_1.$$

The same is true for δ_2 instead of δ_1, which concludes the non-trivial direction of the equality.

Now suppose that $\delta_1^* \vee \delta_2^* \leq \mu \prec (\delta_1 \vee \delta_2)^*$, and repeat the proof of Proposition 9 for $(\delta_1 \wedge \delta_2)$-crossedges. The claim is proved.

Thus $\delta \mapsto \delta^*$ must map any sublattice of N isomorphic to N_5 to a sublattice of $\operatorname{Con} \mathbf{A}$ isomorphic to N_5, and containing only types 2. This is impossible by Theorem 30. □

Note: It is not generally true for (finite) lattices that a $\prec$-preserving $\wedge$-homomorphism is also a lattice homomorphism: consider the distributive lattice $\mathbf{2}^3$. Omitting the element $(1, 1, 0)$, we get a poset that is ordered as a lattice (and isomorphic to the lattice $\mathbf{D}_1$). The inclusion map from this latter lattice to $\mathbf{2}$ is a $\wedge$-homomrphism, preserves $\prec$, but it is not a $\vee$-homomorphism, as $(1, 0, 0) \vee (0, 1, 0)$ is $(1, 1, 0)$ in the distributive lattice, and $(1, 1, 1)$ in the other.

Problem 32. Is Theorem 30 true for quasiorders? If not, is it true if the algebra generates a variety omitting 1?

6. Varieties omitting certain types

Lemma 33. *For any $i \in \{1,2,3,4,5\}$ and any variety $\mathcal{V}$, $\mathcal{V}$ omits i for congruences iff it omits i for quasiorders.*

Proof. Suppose that $\mathbf{A}$ is a finite algebra in $\mathcal{V}$, (α, β) is a prime quasiorder quotient of $\mathbf{A}$ with type i. If (α, β) is a *-quotient, then $\mathrm{typ}(\alpha^*, \beta^*) = i$, so $\mathcal{V}$ does not omit i for congruences.

Now assume that (α, β) is a +-quotient. By definition, its type is determined by types of congruence quotients in the enlargement $\mathbf{M}_+$ of an (α, β)-minimal set M (for the sake of simplicity, we introduce $\mathbf{M} = \mathbf{A}|_M$).

There is an idempotent unary polynomial e mapping A onto M such that $e(\beta) \nsubseteq \alpha$. If $e(x) = t(x, a_1, \ldots, a_k)$, where t is a term of $\mathbf{A}$ and $a_1, \ldots, a_k \in A$, we define $e'(x) := t(x, (a_1, a_1, a_1), \ldots, (a_k, a_k, a_k))$, which is an idempotent unary polynomial mapping $\mathbf{A}_+$ onto $\mathbf{M}_+$. (Both $\mathbf{A}_+$ and $\mathbf{M}_+$ are meant as enlargements by β.)

Consider the restriction mapping from $\mathrm{Con}\,\mathbf{A}_+$ to $\mathrm{Con}\,\mathbf{M}_+$. As $M_+ = e'(A_+)$, this mapping is onto by [HM 2.3.] (MAYBE THERE IS A BETTER THEOREM HERE SO THE NEXT IS USELESS.), more precisely, for any congruence $\mu \in \mathrm{Con}\,\mathbf{M}_+$, the restriction of the congruence $\mu_0 \in \mathrm{Con}\,\mathbf{A}_+$ generated by μ to M_+ is μ. Therefore, for any prime quotient (μ, ν) of $\mathrm{Con}\,\mathbf{M}_+$, there are congruences $\mu_0 \leq \mu_1 \prec \nu_1 \leq \nu_0$ of $\mathrm{Con}\,\mathbf{A}_+$ such that the restriction of μ_1 and ν_1 to M_+ is μ and ν, respectively. But then, a minimal (μ_1, ν_1)-set of $\mathbf{A}_+$ contained in M_+ (there is such a minimal set, for M_+ is a polynomial image of A_+) is also a minimal (μ, ν)-set of M_+. Thus $\mathrm{typ}(\mu_1, \nu_1) = \mathrm{typ}(\mu, \nu)$, so each type in the congruence lattice of $\mathbf{M}_+$ (including $\mathrm{typ}(\alpha, \beta)$) is present at the congruence lattice of $\mathbf{A}_+$. As $\mathbf{A}_+ \in \mathcal{V}$, i is not omitted by $\mathcal{V}$ for congruences in this case, either. ☐

Theorem 34. *Suppose* $\mathbf{A}$ *generates a variety that omits 1 (omits 1 and 2) for quasiorders. If* $\mathbf{M}$ *is a simple sublattice of* $\mathrm{Quo}\,\mathbf{A}$*, then it is modular lattice (a two-element lattice).*

Proof. We can assume that $\mathbf{A}$ is $(0_M, 1_M)$-minimal, because if $0_M \leq \gamma_1 \prec \gamma_2 \leq 1_M$ in $\mathrm{Quo}\,\mathbf{A}$, and U is a (γ_1, γ_2)-minimal set, then $\mathbf{A}|U$ generates a variety omitting 1, and its quasioder lattice contains a sublattice isomorphic to a nontrivial homomorphic image of $\mathbf{M}$. (Because $\delta \mapsto \delta|_U$ is a lattice homomorphism from $\mathrm{Quo}\,\mathbf{A}$ to $\mathrm{Quo}\,\mathbf{A}|_U$ which does not collapse γ_1 and γ_2, and $\mathbf{M}$ is simple.)

By Theorem 29, the interval $[0_M, 1_M]$ is either a distributive lattice, or it only contains types 2. In the latter case, the interval is modular by Theorem 31, thus M is also modular as its sublattice. If the variety omits 2 as well as 1, then M is a simple distributive lattice, hence it is a two-element lattice. ☐

THE ANSWER TO THE FOLLOWING IS NO (5- OR 6-GENERATED FREE SEMILATTICE).

Problem 35. Is it true that in a variety omitting 1, all the quasiorder lattices are $\wedge$-semidistributive over modular, i.e they have a congruence such that the lattice is $\wedge$-semidistributive factorised by this congruence, and the congruence classes are all modular lattices. This possible generalization of [9.6] is stronger, than the preceding theorem. On the other hand, it is most likely not true (possible counterexample: a sufficiently ugly semilattice).

The aim of the remainder of this section is to generalize ?????'s result [CITATION] stating that in a variety omitting 1, 2, and 5, the congruence latties of the finite algebras are not only $\vee$-semidistributive ([CITATION]), but lower bounded. By the previous theorem, there is no confusion about whether we talk about omitting *congruence types* ot omitting *quasiorder types.*

Before we can prove lower boundedness, we first need to prove $\vee$-semidistributivity.

Lemma 36. *Suppose that C is a finite set, $\mathbf{L} \leq \operatorname{Pre} C$ is a lattice of preorders on C, and $\beta \in C$. Then the mapping $\delta \mapsto \delta \wedge \beta^{-1}$ is a lattice homomorphism from the ideal $(\beta]$ of $\mathbf{L}$ to the ideal $(\beta^*]$.*

Proof. It is obvious that this mapping preserves meets. We need to show that for any $\delta_1, \delta_2 \leq \beta$:

$$(\delta_1 \vee \delta_2) \wedge \beta^{-1} \subseteq (\delta_1 \wedge \beta^{-1}) \vee (\delta_2 \wedge \beta^{-1}).$$

Consider an edge (a, b) from the left hand side. There must be elements of C $a = c_0, c_1, \ldots, c_k = b$ so that for each $0 \leq i < k$, (c_1, c_{i+1}) is in either δ_1 or δ_2. As

$$a = c_0 \beta c_1 \beta \ldots \beta c_k = b \beta a,$$

all the c_i are in the same β^*-block, so each (c_i, c_{i+1}) is in either $\delta_1 \wedge \beta^{-1}$ or $\delta_2 \wedge \beta^{-1}$. □

OMIT THE ABOVE PROOF IF IT IS IN THE QLAT ARTICLE.

Theorem 37. *Suppose that $\mathbf{A}$ is a finite algebra in a variety omitting 1, 2, and 5. Then $\operatorname{Quo} \mathbf{A}$ is $\vee$-semidistributive.*

Proof. Suppose that $\alpha, \gamma_1, \gamma_2 \in \operatorname{Quo} \mathbf{A}$ are such that $\alpha \vee \gamma_1 = \alpha \vee \gamma_2 > \alpha \vee (\gamma_1 \wedge \gamma_2)$. We may assume that $\alpha = \alpha \vee (\gamma_1 \wedge \gamma_2)$, and that $\beta := \alpha \vee \gamma_1 \succ \alpha$. Suppose that $\mathbf{A}$ is such that $|A|$ is minimal among counterexamples to the theorem, and the quasiorders are such that β is minimal.

As a variety omitting 1, 2, and 5 is a property characterised by idempotent identities [CITATION], we can deduce from the minimal cardinality of $\mathbf{A}$ that $\mathbf{A}$ is (α, β)-minimal. We will consider two cases depending on whether it is a *-quotient or a +-quotient.

If $\alpha^* \neq \beta^*$, β must be a congruence, otherwise intersecting $\alpha, \gamma_1, \gamma_2$ with β^{-1} would yield a counterexample contradicting the minimality of β (see Lemma 36). As $\mathbf{A}$ is (α, β)-minimal, it also is (α^*, β)-minimal. By Proposition 5, (α^*, β) is a prime congruence quotient. As $\operatorname{typ}(\alpha^*, \beta) \in \{3, 4\}$, there is a unique two-element (α^*, β)-body (say, $\{u, v\}$). Either (u, v) or (v, u) is not in α. That edge must be in γ_1, as it is in $\beta = \alpha \vee \gamma_1$, and $\{u, v\}$ is a β-block. Similarly, it must be in γ_2, which contradicts $\gamma_1 \wedge \gamma_2 \leq \alpha$.

Now assume $\alpha^* = \beta^*$. In this case, by Theorem 24, $\beta \backslash \alpha$ must be a single edge. This is impossible: by $\alpha \vee \gamma_1 = \alpha \vee \gamma_2 = \beta$, both γ_1 and γ_2 must contain this edge, but then $\gamma_1 \wedge \gamma_2 \not\leq \alpha$. □

Lemma 38. *Suppose $\mathbf{L}$ is a lattice, and θ is a congruence on $\mathbf{L}$ such that all θ-blocks have at most two element. Then if $\mathbf{L}$ is join semidistributive, then so is $\mathbf{L}/\theta$.*

Proof. Suppose there are elements $a, b, c \in L$ so that

$$(a \vee b)/\theta = (a \vee c)/\theta > (a \vee (b \wedge c))/\theta.$$

By the join semidistributivity of $\mathbf{L}$, $a \vee b = a \vee c$ is impossible, but $a \vee b$ and $a \vee c$ are in the same (necessarily two-element) θ-block. We may assume that $a \vee c \prec a \vee b$.

We claim that

$$a \vee ((a \vee c) \wedge b) = a \vee c.$$

It is obvious that $\leq$ is satisfied in the above equality, but also, the two sides are in the same θ-block by

$$a \vee ((a \vee c) \wedge b) \theta a \vee ((a \vee b) \wedge b) = a \vee b \theta a \vee c,$$

and $a \vee c$ is the smallest element of that θ-block.

By using the join semidistributivity of $\mathbf{L}$, we gain

$$a \vee c = a \vee (c \wedge (a \vee c) \wedge b) = a \vee (b \wedge c),$$

which contradicts $(a \vee c)/\theta > (a \vee (b \wedge c))/\theta$. □

Lemma 39. *Suppose* $\mathbf{L}$ *is a finite lattice, and* θ *is a congruence on* $\mathbf{L}$ *such that all* θ*-blocks have at most two elements. If* $\mathbf{L}$ *is join semidistributive and* $\mathbf{L}/\theta$ *is lower bounded, then* $\mathbf{L}$ *is also lower bounded.*

Proof. Let $\mathbf{L}' := \mathbf{L}/\theta$, and denote with K the set of elements of L' corresponding to two-element θ-classes. For any $k \in K$, there are elements $k_b \prec k_t$ of L such that $k_b/\theta = k_t/\theta = k$. We introduce the binary relation $\rightsquigarrow$ on K with

$$k^{(1)} \rightsquigarrow k^{(2)} \Leftrightarrow (k_t^{(1)} \vee k_b^{(2)} = k_t^{(2)}) \wedge (k_t^{(1)} \wedge k_b^{(2)} = k_b^{(1)}),$$

which is a partial order on K. We denote with $\sim$ the equivalance relation generated by it.

Claim 1. $\sim = \rightsquigarrow \circ \rightsquigarrow^{-1}$.

As $\rightsquigarrow$ is a partial order, it is enough to show $\rightsquigarrow^{-1} \circ \rightsquigarrow \subseteq \rightsquigarrow \circ \rightsquigarrow^{-1}$. Suppose that $k^{(1)} \rightsquigarrow k^{(2)} \rightsquigarrow^{-1} k^{(3)}$. By the join semidistributivity of $\mathbf{L}$ and

$$k_b^{(2)} \vee k_t^{(1)} = k_b^{(2)} \vee k_t^{(3)} = k_t^{(2)},$$

we deduce that $k_t^{(1)} \wedge k_t^{(3)} \not\leq k_b^{(2)}$, thus

$$k_t^{(1)} \wedge k_t^{(3)} \neq k_b^{(1)} \wedge k_b^{(3)}.$$

This means that $k^{(1)} \wedge k^{(3)} \in K$, and it is easy to check that $k^{(1)} \rightsquigarrow^{-1} k^{(1)} \wedge k^{(3)} \rightsquigarrow k^{(3)}$. The claim is proved.

Claim 2. *For* $k^{(1)}, k^{(2)} \in K$, $k^{(1)} \sim k^{(2)}$ *is equivalent to* $k^{(1)} \rightsquigarrow^{-1} k^{(1)} \wedge k^{(2)} \rightsquigarrow k^{(2)}$.

By the previous claim, we only have to prove that if $k^{(1)} \rightsquigarrow^{-1} k^{(3)} \rightsquigarrow k^{(2)}$, then we can exchange $k^{(3)}$ into $k^{(1)} \wedge k^{(2)}$. By $k_b^{(3)} \leq k_b^{(1)}, k_b^{(2)}$, we get that $k^{(3)} \leq k^{(1)} \wedge k^{(2)}$. But $k_t^{(3)} \not\leq k_b^{(1)}, k_b^{(2)}$, which is only possible if $k_b^{(1)} \wedge k_b^{(2)}$ is the bottom element of a two-element θ-class, whose top element is not lower than either $k_b^{(1)}$ or $k_b^{(2)}$. This means precisely that $k^{(1)} \rightsquigarrow^{-1} k^{(1)} \wedge k^{(2)} \rightsquigarrow k^{(2)}$.

Claim 3. *The* $\sim$*-classes are lower pseudointervals in* L'.

The previous claim shows that they are closed to intersection, all we need to prove that they are convex. Suppose that $k^{(1)} \leq l' \leq k^{(2)}$, where $l' \in L'$, and $k^{(1)} \sim k^{(2)}$. Again by the previous claim, we get that $k^{(1)} \rightsquigarrow k^{(2)}$. Hence, $l' \in K$, because otherwise $k_t^{(1)}$ would be smaller then the (unique) element of L corresponding to l', which would be smaller then $k_b^{(2)}$. Now if $k^{(1)} \not\rightsquigarrow l'$, then $k_t^{(1)} \leq l'_b$, but then also $k_t^{(1)} \leq k_b^{(2)}$, a contradiction. Therefore, l' is in the $\sim$-class of $k^{(1)}$, and the claim is proved.

As the $\sim$-classes are obviously disjoint, we only have to prove that if we double the $\sim$-classes in $\mathbf{L}'$, we get a lattice isomorphic to $\mathbf{L}$. To do that, check that for $k^{(1)} \sim k^{(2)}$,

$$k_b^{(1)} \leq k_b^{(2)} \Leftrightarrow k^{(1)} \leq k^{(2)} \Leftrightarrow k_t^{(1)} \leq k_t^{(2)} \Leftrightarrow k_b^{(1)} \leq k_t^{(2)},$$

and $k_t^{(1)} \leq k_b^{(2)}$ is impossible (by Claim 2), and if for $l_1, l_2 \in L$, $l_1/\theta \not\sim l_2/\theta$, then

$$l_1 \leq l_2 \Leftrightarrow l_1/\theta \leq l_2/\theta.$$

□

Lemma 40. *Suppose $\mathbf{L}$ is a finite join semidistributive lattice, and θ is a congruence on $\mathbf{L}$ such that all θ-blocks having at least three elements are isomorphic to the direct square of the two-element lattice. If θ is an atom in $\operatorname{Con}\mathbf{L}$, then no block of it contains more than two elements.*

Proof.

Claim 1. *Suppose that $a \prec b$ with a and b being in the same θ-block. Then for $a \vee c \preceq b \vee c$ and $a \wedge c \preceq b \wedge c$ for any $c \in L$.*

Assume first that $a \vee c < x, y < b \vee c$ (the case when $a \vee c$ and $b \vee c$ is not in a one- or two-element θ-block). As neither x nor y can be larger than (or equal to) b,

$$x \vee y = x \vee b = b \vee c > x = x \vee a = x \vee (y \wedge b),$$

contradicting join semidistributivity. Likewise, if $a \wedge c < x, y < b \wedge c$, then neither x nor y can be smaller than (or equal to) a, and

$$a \vee x = a \vee y = b > a = a \vee (a \wedge c) = a \vee (x \wedge y)$$

contradicts join semidistributivity. The claim is proved.

Introduce the relation $\rightsquigarrow$ on the prime quotients of $\mathbf{L}$ with

$$(x_1, x_2) \rightsquigarrow (y_1, y_2) \Leftrightarrow (x_2 \vee y_1 = y_2) \wedge (x_2 \wedge y_1 = x_1),$$

this is a preorder.

Claim 2. *The transitive closure of this relation is $\sim := \rightsquigarrow \circ \rightsquigarrow^{-1}$.*

Suppose that

$$(x_1, x_2) \rightsquigarrow (y_1, y_2) \rightsquigarrow^{-1} (z_1, z_2),$$

let us show that

$$(x_1, x_2) \rightsquigarrow^{-1} (x_1 \wedge z_1, x_2 \wedge z_2) \rightsquigarrow (z_1, z_2).$$

For that, it is enough to prove that $x_1 \wedge z_1 \prec x_2 \wedge z_2$. By the join semidistributivity of $\mathbf{L}$ we get that $x_2 \wedge z_2 \not\leq y_1$, otherwise

$$y_1 \vee x_2 = y_1 \vee z_2 = y_2 > y_1 = y_1 \vee (x_1 \wedge z_1),$$

hence $x_1 \wedge z_1 \neq x_2 \wedge z_2$. By the previous claim,

$$x_1 \wedge z_1 \preceq x_1 \wedge z_2, x_2 \wedge z_1 \preceq x_2 \wedge z_2,$$

so if $x_1 \wedge z_1$ and $x_2 \wedge z_2$ were the respectively the bottom and the top element of a θ-class, the other two elements would have to be $x_1 \wedge z_2$ and $x_2 \wedge z_1$ (because $x_1 \wedge z_2 = x_2 \wedge z_1 > x_1 \wedge z_1$ is clearly not the case). But that would mean that

$$(x_1 \wedge z_2) \vee (x_2 \wedge z_1) = x_2 \wedge z_2,$$

which contradicts $x_2 \wedge z_2 \not\leq y_1$. The claim is proved.

Now assume that there is a θ-block with elements $a < b, c < d$. As θ is an atom in $\operatorname{Con} \mathbf{L}$, it is the congruence generated by the edge (a, b). The set

$$\rho := \{(c, d) \in L^2 : (c = d) \vee ((c, d) \sim (a, b))\}$$

is closed under the unary polynomials of $\mathbf{L}$ by Claim 1, therefore τ is the transitive closure of ρ. This is only possible if $(b, d) \in \rho$ (because if $(a, c) \in \rho$, then by $(a, c) \rightsquigarrow (b, d)$ we see that (b, d) is also in ρ).

By Claim 2, there is a prime quotient (e, f) in $\mathbf{L}$ so that $(a, b) \rightsquigarrow^{-1} (e, f)$ and $(e, f) \rightsquigarrow (b, d)$. The first relation implies that $f \leq b$, the second that $f \vee b = d$, this contradiction finishes the proof. $\square$

Lemma 41. *Suppose* $\mathbf{L}$ *is a finite lattice, and* θ *is a congruence on* $\mathbf{L}$ *such that all* θ*-blocks having at least three elements are isomorphic to the direct square of the two-element lattice. If* $\mathbf{L}$ *is join semidistributive and* $\mathbf{L}/\theta$ *is lower bounded, then* $\mathbf{L}$ *is also lower bounded.*

Proof. Suppose that in the congruence lattice of $\mathbf{L}$,

$$0_{\mathbf{L}} \prec \theta_1 \prec \cdots \prec \theta_k = \theta.$$

We use induction on k. If $k = 1$ then we are ready by Lemma 39 and Lemma 40.

If $k > 1$, then by Lemma 38, $\mathbf{L}/\theta_1$ is a join semidistributive lattice. Notice that θ/θ_1 is a congruence of $\mathbf{L}/\theta_1$ that does not have a block with more than two elements that is not isomorphic with the direct square of the two-element lattice. Hence by the inductive hypothesis $\mathbf{L}/\theta_1$ is a lower bounded lattice, and then so is $\mathbf{L}$, as we have already proved the theorem for the $k = 1$ case. $\square$

Theorem 42. *Suppose that* $\mathbf{A}$ *is a finite algebra in a (congruence) join semidistributive variety. Then* $\operatorname{Quo} \mathbf{A}$ *is a lower bounded lattice.*

Proof. We use induction on $|A|$. For any prime quotient $\alpha \prec \beta$ in $\operatorname{Quo} \mathbf{A}$ such that $\mathbf{A}$ is not (α, β)-minimal, take an $(\alpha, beta)$-minimal set M. The algebra $\mathbf{A}|_M$ is in a join semidistributive variety, thus by the inductive hypothesis $\operatorname{Quo} \mathbf{A}|_M$ is a lower bounded lattice. By Propositions 1 and 2, the restriction to M is a lattice homomorphism from $\operatorname{Quo} \mathbf{A}$ to $\operatorname{Quo} \mathbf{A}|_M$. Denote the kernel of this homomorphism with $\theta_{\alpha,\beta}$. Note that α and β are in different $\theta_{\alpha,\beta}$-blocks, and that $\operatorname{Quo} \mathbf{A}/\theta_{\alpha,\beta}$ is a lower bounded lattice (as it is isomorphic to a sublattice of the lower bounded $\operatorname{Quo} \mathbf{A}|_M$).

Introduce

$$\theta := \bigwedge_{\mathbf{A}\text{is not } (\alpha,\beta)\text{-minimal}} \theta_{\alpha,\beta}.$$

As

$$\operatorname{Quo}\mathbf{A}/\theta \leq_S \prod_{\mathbf{A}\text{is not }(\alpha,\beta)\text{-minimal}} \operatorname{Quo}\mathbf{A}/\theta_{\alpha,\beta},$$

we get that $\mathbf{A}/\theta$ is a lower bounded lattice. As for each $\alpha \prec \beta$ in $\operatorname{Quo}\mathbf{A}$ such that $\mathbf{A}$ is not (α, β)-minimal, $(\alpha, \beta) \notin \theta_{\alpha,\beta}$, each θ-block is such that $\mathbf{A}$ is minimal to each prime quotient in it.

Consequently, $\mathbf{A}$ is minimal for any quasiorders $\gamma < \delta$ satisfying $(\gamma, \delta) \in \theta$. By Theorem 29, all θ-blocks have either one or two element, or are isomorphic to the lattice $\mathbf{2}^2$ (as $\mathbf{A}$ is in a variety omitting 1, 2, and 5). By Theorem 37, $\operatorname{Quo}\mathbf{A}$ is a join semidistributive lattice. Thus by Lemma 41, $\operatorname{Quo}\mathbf{A}$ is a lower bounded lattice. $\square$