MATEMATIKA II

2007. 02. 16

Követelmény: szigorlat

Ápr. 14. ZH.

pót ZH az utolsó alkalommal!

aláírás + gyakjegy => Szigorlat)

ütemterv a neten – gyakorló feladatok
szigorlati sorok

Integrál számítás és alkalmazásai

A határozatlan integrál

Def.: A $F(x)$ fgv. a $f(x)$ fgv. primitív fgv-ének nevezzük, ha $F'(x) = f(x)$

$$[F(x) + C]' = \frac{d[F(x)+C]}{dx} = F'(x) + \underbrace{(C)'}_{=0} =$$

$(C \in \mathbb{R})$

$$= F'(x) = f(x) \Rightarrow$$

$\Rightarrow$ ha a f fgv-hez találok egy primitív fgv-t akkor f-hez végtelen sok primitív fgv. tartozik.

pl.: $f(x) = x^4 \Rightarrow F(x) = \frac{x^5}{5}$, mert

$$F'(x) = \left(\frac{x^5}{5}\right)' = \frac{1}{5} 5x^4 = x^4 = f(x)$$

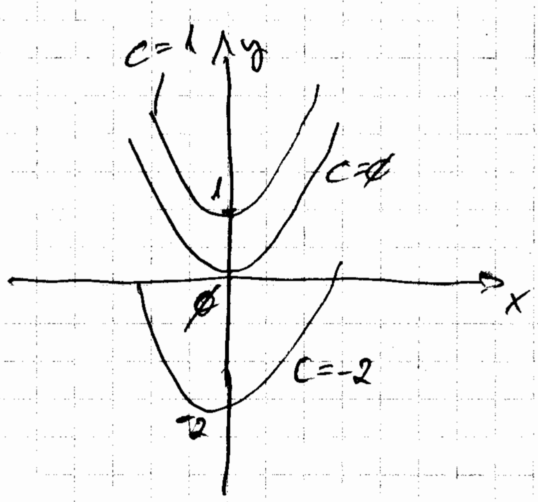
$$F(x) = \frac{x^5}{5} + 6 \; ; \quad F(x) = \frac{\cancel{x^4}^{\,5}}{5} - \pi \; ; \quad \ldots$$

$$F(x) = \frac{x^5}{5} + C \; ; \quad (C \in \mathbb{R})$$

$$\int \underbrace{f(x)}_{\text{integrandusz}} \overbrace{dx}^{\text{jelöli, hogy az integrálás az } x \text{ vált. szerint történik.}} = \underbrace{F(x)}_{\text{prim. fgv.}} + \underbrace{C}_{\text{integrációs állandó}}$$

pl.: $\int x\,dx = \frac{x^2}{2} + C$

mert $\left(\frac{x^2}{2} + C\right)' = \frac{1}{2} \cdot 2x + 0 = \underline{\underline{x}}$



$c = 0 \quad y = x^2/2$

$c = 1 \quad y = \frac{x^2}{2} + 1$

$c = -2 \quad y = \frac{x^2}{2} - 2$

<u>Alapintegrál</u>

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \; ; \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \; ; \quad (x \neq 0)$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \quad ; \quad (a > 0 \,;\, a \neq 1)$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x} dx = \operatorname{tg} x + C$$

$$\int \frac{1}{\sin^2 x} dx = -\operatorname{ctg} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$

$$\int \frac{-1}{\sqrt{1-x^2}} dx = \arccos x + C$$

$$\int \frac{1}{1+x^2} dx = \operatorname{arc\,tg} x + C$$

$$\int \frac{-1}{1+x^2} dx = \operatorname{arc\,ctg} x + C$$

$$\int \operatorname{sh} x \, dx = \operatorname{ch} x + C$$

$$\int \operatorname{ch} x \, dx = \operatorname{sh} x + c$$

$$\int \frac{1}{\operatorname{ch}^2 x} \, dx = \operatorname{th} x + c$$

$$\int \frac{1}{\operatorname{sh}^2 x} \, dx = -\operatorname{cth} x + c$$

Integrálási szabályok: Tegyük fel, hogy az f és g fg-k integrálhatók

1. $\int c f(x)\,dx = c \int f(x)\,dx$; $(c \in \mathbb{R})$

2. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

1. + 2. $\Rightarrow \int [c_1 f(x) + c_2 g(x)]\,dx = c_1 \int f(x)\,dx + c_2 \int g(x)\,dx$

3. $$\int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C \qquad (n \neq -1)$$

4. $$\int \frac{f'(x)}{f(x)}\,dx = \ln |f(x)| + c$$

5. $$\int f(\lambda x)\,dx = \frac{F(\lambda x)}{\lambda} + c$$; ahol $\lambda \in \mathbb{R} \setminus \{0\}$ és

$F'(x) = f(x)$

$$\int (x^3 - \underbrace{\frac{1}{x^4}}_{x^{-4}} + 2e^x - \cos x)\,dx = \frac{x^4}{4} - \frac{x^{-4+1}}{-4+1} + 2e^x - \sin x + c$$

$$\int (3\underbrace{\sqrt[3]{x}\cdot\sqrt{3}}_{x^{\frac{1}{3}}\cdot x^{\frac{1}{2}}} - \frac{1}{x} + 2^x)\,dx = 3\cdot\frac{x^{5/6+1}}{5/6+1} - \ln|x| + \frac{2^x}{\ln 2} + c$$

$$\underbrace{x^{\frac{1}{3}}\cdot x^{\frac{1}{2}}}_{x^{\frac{1}{3}+\frac{1}{2}} = x^{\frac{5}{6}}}$$

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szabály

(3) $\int \sqrt{1-x^2}\,2x\,dx = \int \underbrace{(1-x^2)}_{f}{}^{\frac{1}{2}}\cdot\underbrace{(2x)}_{f'}\,dx = -\frac{(1-x^2)^{3/2}}{3/2} + c$

$(1-x^2)' = -2x$

$f(x) = 1-x^2$ $\quad n = \frac{1}{2}$

$f'(x) = -2x$

$$\int \frac{x}{x^2+5}\,dx = \frac{1}{2}\int\frac{2x}{x^2+5}\,dx = \frac{1}{2}\ln(x^2+5) + c$$

$(x^2+5)' = 2x$

$$\int (\sin x - 5\cdot\cos 2x + \frac{3}{\cos^2 x})\,dx = -\cos x - 5\cdot\frac{\sin 2x}{2} + 3\,\mathrm{tg}\,x + c$$

$(\sin 2x)' = 2\cdot\cos 2x$ $\qquad (2x)' = 2$

azzal azért osztható mert konstans

$$\int \operatorname{tg}^2 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{1-\cos^2 x}{\cos^2 x} \, dx =$$

$$= \int \left(\frac{1}{\cos^2 x} - 1 \right) dx = \operatorname{tg} x - x + C$$

$$\int \frac{1}{\sin x} \, dx = \int \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}} \, dx = \int \left(\frac{\sin \frac{x}{2}}{2 \cos \frac{x}{2}} + \frac{\cos \frac{x}{2}}{2 \sin \frac{x}{2}} \right) dx =$$

$\sin 2\alpha = 2 \sin \alpha \cdot \cos \alpha$

$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

másik oldalon folyt.

$\sin x = \sin 2\frac{x}{2} = 2 \sin \frac{x}{2} \cos \frac{x}{2}$

$$\left.\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ \cos^2 x - \sin^2 x &= \cos 2x \end{aligned}\right\}$$

$(+)$: $2\cos^2 x = 1 + \cos 2x \Rightarrow \boxed{\cos^2 x = \frac{1+\cos 2x}{2}}$

$(-)$: $\cancel{2\cos^2 x}$

$2 \cdot \sin^2 x = 1 - \cos 2x \Rightarrow \boxed{\sin^2 x = \frac{1-\cos 2x}{2}}$

$$\left(\cos \frac{x}{2}\right)' = -\left(\sin \frac{x}{2}\right) \cdot \frac{1}{2} = -\frac{1}{2} \cdot \sin \frac{x}{2}$$

$$\left(\sin \frac{x}{2}\right)' = \left(\cos \frac{x}{2}\right) \frac{1}{2} = \frac{1}{2} \cdot \cos \frac{x}{2}$$

$$= \int \left(\frac{\frac{1}{2} \cdot \sin \frac{x}{2}}{\cos \frac{x}{2}} + \frac{\frac{1}{2} \cdot \cos \frac{x}{2}}{\sin x/2} \right) dx = -\ln \left| \cos \frac{x}{2} \right| + \ln \left| \sin \frac{x}{2} \right| + c =$$

$$= \ln \left| \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right| + c = \ln \left| \operatorname{tg} \frac{x}{2} \right| + c$$

$$\int \frac{\ln x}{x} dx = \int (\ln x) \cdot \frac{1}{x} dx = \frac{(\ln x)^2}{2} + C$$

$f(x) = \ln x \qquad u = 1 \qquad f'(x) = \frac{1}{x}$

$$\int \frac{dx}{x \ln x} = \int \frac{\frac{1}{x}}{\ln x} dx = \ln |\ln x| + c$$

PARCIÁLIS INTEGRÁLÁS

$u(x)$ és $v(x)$ differenciálható fgv.-eket

$$[u(x)\, v(x)]' = u'(x)\, v(x) + u(x)\, v'(x) \quad \Big| \int$$

$$u(x)\, v(x) = \int u'(x)\, v(x)\, dx + \int u(x)\, v'(x)\, dx$$

$$\boxed{\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int u'(x)\, v(x)\, dx}$$

Integrandusban $P_n(x)$ $\left\{ \begin{array}{l} \text{exp.} \\ \text{v.} \\ \text{trig (sin, cos)} \\ \text{v.} \\ \text{hip (sh, ch)} \\ v'(x) \end{array} \right.$

$u(x)$

$$\int x\cos 2x\,dx = \left| \begin{matrix} u = x & v' = \cos 2x \\ u' = 1 & v = \frac{\sin 2x}{2} \end{matrix} \right| =$$

$$= x\cdot\frac{\sin 2x}{2} - \int \frac{\sin 2x}{2}\,dx = \cancel{x}\,\frac{\sin 2x}{2} - \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + C =$$

$$= x\cdot\frac{\sin 2x}{2} + \frac{1}{4}\cos 2x + C\ ;$$

Ell. $\left(x\cdot\frac{\sin 2x}{2} + \frac{1}{4}\cos 2x + C\right)' = x\cdot\cos 2x$

$$\int x^2 e^{-x}\,dx = \left| \begin{matrix} u = x^2 & v' = e^{-x} \\ u' = 2x & v = -e^{-x} \end{matrix} \right| =$$

$$= x^2\cdot(-e^{-x}) - \int 2x(-e^{-x})\,dx = -x^2e^{-x} + \int 2xe^{-x}\,dx =$$

$$= \left| \begin{matrix} u = 2x & v' = e^{-x} \\ u' = 2 & v = -e^{-x} \end{matrix} \right| = -x^2e^{-x} + \left[2x(-e^{-x}) - \int 2\cdot(-e^{-x})\,dx\right] =$$

$$= -x^2e^{-x} - 2xe^{-x} + \int 2\cdot e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} + 2(-e^{-x}) + C$$

$$\int x^2\sin x\,dx\ ;\quad \int x e^{5x}\,dx\ ;\quad \int 4x\,\mathrm{ch}\,5x\,dx$$

2. integrandusban: 1 (beírom) $v'(x)$

log. v. arcusz fgv. v. area } van $u(x)$

$(\ln 3x)' = \frac{1}{3x} \cdot 3 = \frac{1}{x}$

$$\int \ln x\,dx = \int 1 \cdot \ln x\,dx = \left| \begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{array} \right| =$$

$$x \ln x - \int \underbrace{\frac{1}{x} \cdot x}_{=1}\,dx = x \ln x - x + C$$

$$\int \ln 3x\,dx = \int 1 \cdot \ln 3x\,dx = \left| \begin{array}{ll} u = \ln 3x & v' = 1 \\ u' = & v = x \end{array} \right| =$$

$(\ln 3x)' = (\ln 3 + \ln x)' = 0 + \frac{1}{x}$

$$x \ln 3x - \int \frac{1}{x} \cdot x\,dx = x \ln 3x - x + C$$

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(3) integrandusban: $\underbrace{x^2}_{v'(x)} \underbrace{(\log_a x)^m}_{u(x)}$

$2 \neq -1$

$m > 0$ egész ; $a > 0$; $a \neq 1$

$$\int x^5 \underbrace{(\ln x)^2}_{= \ln^2 x} dx = \left| \begin{array}{ll} u = (\ln x)^2 & v' = x^5 \\ u' = 2(\ln x) \cdot \frac{1}{x} & v = \frac{x^6}{6} \end{array} \right| =$$

$$\frac{x^6}{6} (\ln x)^2 - \int 2 \cdot \frac{\ln x}{x} \cdot \frac{x^6}{6}\,dx = \frac{x^6}{6} \cdot \ln^2 x - \frac{1}{3} \int x^5 \ln x\,dx =$$

$$\left| \begin{matrix} u = \ln x & v' = x^5 \\ u' = \frac{1}{x} & v = \frac{x^6}{6} \end{matrix} \right| =$$

$$= \frac{x^6}{6} \cdot \ln^2 x - \frac{1}{3}\left[\frac{x^6}{6}\ln x - \int \frac{1}{x} \cdot \frac{x^6}{6} dx\right] = \frac{x^6}{6}\ln^2 x - \frac{1}{18}x^6 \ln x +$$

$$\cancel{+\frac{1}{18}\frac{x^6}{6}} \quad + \frac{1}{18}\frac{x^6}{6} + C$$

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d, integrandusban : exp., trig. (sin, cos) v/és hip. (sh, ch) fgv. ez szorzatként szerepel.

$$\int e^x \sin 2x\, dx = \left| \begin{matrix} u = e^x & v' = \sin 2x \\ u' = e^x & v = -\frac{\cos 2x}{2} \end{matrix} \right| =$$

$$= -\frac{1}{2}e^x \cdot \cos 2x - \int -\frac{e^x \cos 2x}{2} dx = -\frac{1}{2} \cdot e^x \cdot \cos 2x + \frac{1}{2}\int e^x \cos 2x\, dx =$$

$$= \left| \begin{matrix} u = e^x & v' = \cos 2x \\ u' = e^x & v = \frac{\sin 2x}{2} \end{matrix} \right| =$$

$$= -\frac{1}{2} \cdot e^x \cos 2x + \frac{1}{2}\left[e^x \cdot \frac{\sin 2x}{2} - \int e^x \cdot \frac{\sin 2x}{2} dx\right] + C$$

$$= \underline{-\frac{1}{2} \cdot e^x \cdot \cos 2x + \frac{1}{4} \cdot e^x \sin 2x - \frac{1}{4}\int e^x \sin 2x\, dx + C}$$

$$\frac{5}{4}\cdot\int e^x \sin 2x\,dx = -\frac{1}{2}e^x \cos 2x + \frac{1}{4} e^x \sin 2x + C \Rightarrow$$

$$\Rightarrow \int e^x \sin 2x\,dx = \frac{4}{5}\left[-\frac{1}{2}e^x \cos 2x + \frac{1}{4} e^x \sin 2x + C\right]$$

$$\int f(x)\,dx = \left| \begin{array}{l} x = \varphi(t), \text{ahol a } \varphi(t) \text{ szig. mon. és v. cs. deriválható} \\ dx = \varphi'(t)\,dt \end{array} \right| =$$

$$= \int f(\varphi(t))\,\varphi'(t)\,dt$$

A helyettesítést fordított irányban is el lehet végezni:

$$\int f(\varphi(t))\cdot\varphi'(t)\,dt = \left| \begin{array}{l} x = \varphi(t) \\ dx = \varphi'(t)\,dt \end{array} \right| = \int f(x)\,dx$$

$$\int (3x-4)^5\,dx = \left| \begin{array}{l} 3x-5=t \\ 3dx=dt \\ dx=\frac{1}{3}dt \end{array} \right| = \int t^5 \frac{1}{3}dt = \frac{1}{3}\int t^5 dt = \frac{1}{3}\cdot\frac{t^6}{6} + C =$$

$$= \underline{\underline{\frac{1}{18}(3x-5)^6 + C}}$$

$$\int (3x-4)^5 dx = \frac{1}{3}\int (3x-4)^5\cdot 3\,dx = \frac{1}{3}\,\frac{(3x-4)^6}{6} + C =$$

$f(x) = 3x-4$; $f'(x) = 3$ $n = 5$

$$= \underline{\underline{\frac{(3x-4)^6}{18} + C}}$$

$$\int \sin(3-x)\,dx = \left| \begin{array}{l} 3-x=t \\ -dx=dt \end{array} \right| = \int \sin t\,(-dt) = \int -\sin t\,dt =$$

$$= \cos t + C = \underline{\underline{\cos(3-x)+C}}$$

$$\int \sin(3-x)\,dx = -\frac{\cos(3-x)}{-1} + C$$

$$\int \frac{dx}{x^2+9} = \frac{1}{9}\int \frac{1}{\frac{x^2}{9}+1}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^3+1}\,dx = \left| \begin{array}{l} \frac{x}{3}=t \\ \frac{1}{3}dx=dt \\ dx=3dt \end{array} \right.$$

$$= \frac{1}{9}\int \frac{1}{t^2+1}\,3dt = \frac{1}{3}\cdot \operatorname{arctg} t + C =$$

$$= \frac{1}{3}\cdot \operatorname{arctg}\frac{x}{3} + C$$

$$\text{vagy} = \frac{1}{9}\,\frac{\operatorname{arctg}\frac{x}{3}}{1/3} + C$$

$$\frac{x}{3} \cdot \frac{1}{3}$$

$$\int e^{\sin x}\cos x\,dx = \int \left(e^{\sin x}\right)'dx = e^{\sin x} + C$$

$$\int \underbrace{e^{\sin x}}_{e^t}\underbrace{\cos x\,dx}_{dt} = \left| \begin{array}{l} \sin x = t \\ \cos x\,dx = dt \end{array} \right| = \int e^t\,dt = e^t + C =$$

$$= \underline{\underline{e^{\sin x}+C}}$$

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = \left| \begin{array}{l} \sqrt{x} = t \\ \frac{1}{2\sqrt{x}}dx = dt \\ dx = 2t\,dt \end{array} \right| = \int \frac{t}{1+t}\, 2t\,dt = \int \frac{2t^2}{1+t}\,dt =$$

$$= 2\cdot\int \frac{t^2}{t+1}\,dt = 2\int \frac{t^2-1+1}{t+1}\,dt$$

racionális tört fgv. áltört

a, polinom osztás

b, trükkös osztás

$$2\int \frac{(t^2-1)+1}{t+1}\,dt = 2\int \left(t-1+\frac{1}{t+1}\right)dt =$$

$$t^2-1 = (t+1)\cdot(t-1)$$

$$\to 2\cdot\left[\frac{t^2}{2} - t + \ln|1+t|\right] + C = 2\cdot\left[\frac{x}{2} - \sqrt{x} + \ln|1+\sqrt{x}|\right] + C$$

Racionális tört fgv-ek integrálása

$$\int \frac{P_n(x)}{Q_m(x)}\,dx$$

Ha $n \geq m$, akkor áltört fgv. ekkor __létezik__ valódi tört fgv.-t felbontani az alábbi módon.

$$\frac{P_n(x)}{Q_m(x)} = \underbrace{r_{n-m}(x)}_{\text{rac. egész fgv.}} + \underbrace{\frac{s_\ell(x)}{Q_m(x)}}_{\text{valódi rac. tört. fgv.}} \quad (\emptyset \leq \ell < m) . \quad m, \ell, n, \text{ poz. egész}$$

polinom osztással vagy nevezetes azonosság alkalmazásával

2, Ha $n < m$, akkor a valódi rac. tört. fgv. előállítható

$$\frac{A}{(x-a)^k} \quad \text{és} \quad \frac{Bx+C}{(x^2+bx+c)^l}$$ alakú törtek összegeként.

a, A nevező elsőfokú : $\int \frac{A}{bx-a}\,dx = \frac{A}{b}\int \frac{1}{x+\frac{a}{b}}\,dx =$
$(b \neq \emptyset)$

$= \frac{A}{b} \ln\left|x+\frac{a}{b}\right| + C$

$$\int \frac{3}{7x+5}\,dx = 3\int \frac{1}{7x+5}\,dx = \frac{3}{7}\int \frac{7}{7x+5}\,dx = \frac{3}{7}\ln|7x+5| + C$$

b., A nevező másodfokú

$$\int \frac{x-2}{x^2-4x}\,dx = \frac{1}{2}\int \frac{2x-4}{x^2-4x}\,dx = \frac{1}{2}\cdot\ln|x^2-4x| + c$$

4-es int. szab. alkalmazható? Igen

$(x^2-4x)' = 2x-4$

A nevező ax^2+bx+c alakú, ahol $a \neq \emptyset$

I. Ha $D = b^2-4ac \not< \emptyset$, akkor az $ax^2+bx+c=\emptyset$

Ebben az esetben, ha a számláló konstans akkor a nevezőt teljes négyzetté kell kiegészíteni

$$\int \frac{dx}{x^2+2x+5} = \int \frac{1}{(x+1)^2+4}\, dx = \frac{1}{4}\int \frac{1}{\left(\frac{x+1}{2}\right)^2+1}\, dx =$$

$$= \frac{1}{4}\, \frac{\operatorname{arc\,tg} \frac{x+1}{2}}{\frac{1}{2}} + C = \frac{1}{2} \operatorname{arc\,tg} \frac{x+1}{2} + C$$

$$\left(\frac{x+1}{2}\right)' = \left(\frac{1}{2}\right)$$

vagy $\frac{x+1}{2} = t$

II $D = b^2 - 4ac = 0$

1db valós kétszeres gyöke van

mivel a nevező egy teljes négyzet konstans szorzóló esetén könnyen integrálható

$$\int \frac{dx}{x^2-6x+9} = \int \frac{1}{(x-3)^2}\, dx = \int (x-3)^{-2}\, dx =$$

$$= \frac{(x-3)^{-2+1}}{-2+1} + C = -\frac{1}{x-3} + C$$

III. $D = b^2 - 4ac > 0$

két különböző valós gyöke van.

ekkor az int. résztörtekre való bontással kezdődik.

$$\int \frac{dx}{ax^2+bx+c} =$$

$ax^2+bx+c=0$, ha $D>0 \Rightarrow x_1 = a_1$
$x_2 = a_2$

$$ax^2+bx+c = a\cdot(x-a_1)\cdot(x-a_2)$$

$$\int \frac{dx}{ax^2+bx+c} = \int \frac{dx}{a(x-a_1)\cdot(x-a_2)} = \int\left(\frac{A}{x-a_1}+\frac{B}{x-a_2}\right)dx$$

$$\int \frac{x}{x^2+x-6}\,dx = \frac{1}{2}\int\frac{2x}{x^2+x-6}\,dx = \frac{1}{2}\int\frac{2x+1-1}{x^2+x-6}\,dx =$$

$$(x^2+x-6)' = 2x+1$$

$$\Rightarrow \frac{1}{2}\int\left(\frac{2x+1}{x^2+x-6}-\frac{1}{x^2+x-6}\right)dx =$$

lehetőleg nem így!!

$$x^2+x-6=0$$

$$x_{1,2} = \frac{-1\pm\sqrt{1+24}}{2} = \frac{-1\pm 5}{2} = \begin{cases} 2 \\ -3 \end{cases}$$

$$\frac{x}{x^2+x-6} = \frac{x}{(x-2)(x+3)} = \frac{A}{x-2}+\frac{B}{x+3} = \frac{A(x+3)+B(x-2)}{(x-2)(x+3)}$$

$$x \equiv A(x+3)+B(x-2)$$

ha $x=2$ | $2 = 5A \Rightarrow A = \frac{2}{5}$

ha $x=-3$ | $-3 = -5B \Rightarrow B = \frac{3}{5}$

$$\int \frac{x}{x^2+x-6}\,dx = \int\left(\frac{2/5}{x-2} + \frac{3/5}{x+3}\right)dx =$$

$$\frac{2}{5}\ln|x-2| + \frac{3}{5}\ln|x+3| + C$$

$$\cancel{\int \frac{x+1}{x^2(x-1)}\,dx = \int\left(\frac{-2}{x} - \frac{1}{x^2} + \frac{2}{x-1}\right)dx = -2\ln|x| - \frac{x^{-1}}{-1} + 2\ln|x-1|}$$

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax\cdot(x-1)+B(x-1)+Cx^2}{x^2\cdot(x-1)}$$

$$x+1 \equiv Ax(x-1) + B(x-1) + Cx^2$$

ha $x=\emptyset$, $1=-B \Rightarrow \boxed{B=-1}$

ha $x=1$, $\boxed{2=C}$

x^2 $\quad \underline{\emptyset = A+C} \Rightarrow \boxed{A=-2}$

$$\int \frac{x+1}{x^2(x-1)}\,dx = \int\left(-\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1}\right)dx =$$

$$-2\ln|x| - \frac{x^{-1}}{-1} + 2\cdot\ln|x-1| + C = \underline{\ln\left(\frac{x-1}{x}\right)^2 + \frac{1}{x} + C}$$

$$\int \frac{3}{x(x^2+1)}\,dx =$$

$$\frac{A}{(x-a)^2} \quad \text{és} \quad \frac{Bx+C}{(x^2+bx+c)^e}$$

$$b^2-4c < \emptyset$$

$$\frac{3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} =$$

$$x^2+1=\emptyset \Rightarrow x^2=-1$$

$$= \frac{A(x^2+1)+x(Bx+C)}{x\cdot(x^2+1)}$$

$$x^2 = i^2$$

$$x_{1,2} = \pm i$$

$$3 \equiv A(x^2+1) + Bx^2 + Cx$$

x^2: $\emptyset = A + B$

x: $\emptyset = C$

x^0: $3 = A$ $\quad$ $B = -3$

$$\int \frac{3}{x(x^2+1)}\,dx = \int \left(\frac{3}{x} + \frac{-3x}{x^2+1} \right) dx =$$

$$= 3\ln|x| - \underbrace{3 \cdot \frac{1}{2}}_{\frac{3}{2}} \ln(x^2+1) + C$$

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d, linearizáló formula használata

$$\boxed{\cos^2 x = \frac{1+\cos 2x}{2} \;;\; \sin^2 x = \frac{1-\cos 2x}{2}}$$

pl:

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx =$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2} \right] + C$$

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx =$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2} \right] + C$$

$$\int \cos^2\frac{x}{3}\,dx = \int \frac{1+\cos\frac{2x}{3}}{2}\,dx =$$

$$= \frac{1}{2}\int\left(1+\cos\frac{2x}{3}\right)dx = \frac{1}{2}\left[x+\frac{\sin\frac{2x}{3}}{2/3}\right]+C$$

$$\left(\frac{2x}{3}\right)' = \frac{2}{3}$$

$$\cancel{\int \sin^2 6x\,dx}$$

$$\int \sin^2 6x\,dx = \int \frac{1-\cos 12x}{2}\,dx =$$

$$= \frac{1}{2}\int(1-\cos 12x)\,dx = \frac{1}{2}\left[x-\frac{\sin 12x}{12}\right]+C$$

2.) $\int(\sin x)^k(\cos x)^l\,dx$

ha k és az l pozitív egész kitevők közül az egyik páratlan, akkor abból a tényezőből leválasztunk egy első hatványt és a megmaradt páros kitevőjű tényezőt

$\sin^2x+\cos^2x=1$ azonossággal átalakítjuk.

$$\int \sin^3x\,dx = \int \sin x\cdot\underbrace{\sin^2x}_{1-\cos^2x}\,dx = \int \sin x(1-\cos^2x)\,dx =$$

$$= \int(\sin x-\sin x\cos^2x\,dx) = -\cos x+\frac{\cos^3x}{3}+C$$

$f(x)=\cos x \qquad n=2$

$f'(x)=-\sin x$

$$\int \sin^5 x \cos^3 x\,dx = \int \sin x \cdot \sin^4 x \cos^3 x\,dx =$$

$$\int \sin x\,(1-\cos^2 x)^2 \cos^3 x\,dx =$$

$$\int \sin^5 x \cdot \cos x \cdot \overbrace{\cos^2 x}^{1-\sin^2 x}\,dx = \int (\cos x \sin^5 x - \cos x \cdot \sin^7 x)\,dx =$$

$n=7$

$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C$$

$f(x) = \sin x$

$f'(x) = \cos x$

$n = 5$

ha k és l páros akkor először linearizáló formulát célszerű alkalmazni.

$$\int \sqrt{a^2 - x^2}\,dx = \left| \begin{array}{l} x = a \cdot \sin t \\ dx = a \cdot \cos t\,dt \end{array} \right| =$$

$\sin^2 t + \cos^2 t = 1 \quad / \cdot a^2$

$a^2 \cdot \sin^2 t + a^2 \cdot \cos^2 t = a^2$

$$a^2 \cos^2 t = a^2 - \underbrace{a^2 \cdot \sin^2 t}_{x^2}$$

$$\int \underbrace{\sqrt{\underbrace{a^2 - a^2 \cdot \sin^2 t}_{a^2 \cdot \cos^2 t}}}_{a \cdot \cos t}\,\cancel{dt} \cdot a \cdot \cos t\,dt = \int a^2 \cdot \cos^2 t\,dt =$$

$$\cancel{a^2 \int 1 + \cos^2 t\,dt} = a^2 \int \frac{1 + \cos 2t}{2}\,dt =$$

$$\int \sqrt{4-x^2}\, dx = \left| \begin{array}{l} x = 2\cdot \sin t \\ dx = 2\cos t\, dt \\ \frac{x}{2} = \sin t \\ t = \arcsin \frac{x}{2} \end{array} \right| =$$

$$= \int \sqrt{\underbrace{4 - 4\sin^2 t}_{\underset{2\cos t}{4\cos^2 t}}}\; 2\cos t\, dt = \int 4\cos^2 t\, dt =$$

$$\frac{\sin 2t}{2} = \frac{2\cdot \sin t \cdot \cos t}{2}$$

$$= 4\int \frac{1+\cos 2t}{2}\, dt = 2\left[t + \frac{\sin 2t}{2}\right] + C$$

$$= 2\left[\arcsin \frac{x}{2} + \frac{x}{2}\sqrt{1-\left(\frac{x}{2}\right)^2}\right] + C$$

$$= \sin t \sqrt{1-\sin^2 t} =$$

$$= \frac{x}{2}\sqrt{1-\left(\frac{x}{2}\right)^2}$$