2007. 02. 24.

$$\int\sqrt{a^2+x^2}\,dx = \left|\begin{matrix} x = a\,sh\,t \\ dx = a\cdot ch\,t\,dt\end{matrix}\right| = \int \underbrace{\sqrt{\underbrace{a^2+a^2sh^2t}_{a^2ch^2t}}}_{a\,ch\,t}\,a\,ch\,t\,dt =$$

$$ch^2t - sh^2t = 1 \quad \Big/\cdot a^2$$

(*) $a^2ch^2t - a^2sh^2t = a^2$

$a^2ch^2t = a^2 + a^2sh^2t$

$$= \int a^2ch^2t\,dt = a^2\int\frac{1+ch2t}{2}\,dt = \ldots$$

$$ch^2t - sh^2t = 1$$

$$ch^2t + sh^2t = ch2t$$

(+) $2ch^2t = 1 + ch2t \Rightarrow \boxed{ch^2t = \frac{1+ch2t}{2}}$

(−) $2sh^2t = ch2t - 1 \Rightarrow \boxed{sh^2t = \frac{ch2t-1}{2}}$

$$\int\sqrt{x^2-a^2}\,dx = \left|\begin{matrix} x = a\,ch\,t \\ dx = a\,sh\,t\,dt\end{matrix}\right| = \int \underbrace{\sqrt{\underbrace{a^2ch^2t - a^2}_{a^2sh^2t}}}_{a\,sh\,t}\,a\,sh\,t\,dt =$$

(*) $a^2ch^2t - a^2 = a^2sh^2t$

$$= \int a^2sh^2t\,dt = a^2\int\frac{ch2t-1}{2}\,dt = \ldots$$

$$\int\sqrt{4+x^2}\,dx = \left|\begin{matrix} x = 2\,sh\,t \\ dx = 2ch\,t\,dt \\ \frac{x}{2} = sh\,t \quad t = arsh\frac{x}{2}\end{matrix}\right| = \int \underbrace{\sqrt{\underbrace{4+4sh^2t}_{4ch^2t}}}_{2ch\,t}\,2ch\,t\,dt =$$

$$= \int \overbrace{2ch\,t\cdot 2\,ch\,t}^{4ch^2t}\,dt = 4\int\frac{1+ch2t}{2}\,dt = 2\left[t + \frac{sh2t}{2}\right] + C =$$

$$= 2\left[\operatorname{arsh}\frac{x}{2} + \frac{x}{2}\sqrt{1+\left(\frac{x}{2}\right)^2}\right] + C$$

$$\frac{\operatorname{sh}2t}{2} = \frac{\cancel{2}\operatorname{sh}t\operatorname{ch}t}{\cancel{2}} = \operatorname{sh}t\sqrt{1+\operatorname{sh}^2 t} = \frac{x}{2}\sqrt{1+\frac{x^2}{4}}$$

$$\cancel{\int\sqrt{2+x^2}\,dx =}$$

$$\int\sqrt{x^2-2}\,dx = \left|\begin{array}{l} x = \sqrt{2}\operatorname{ch}t \\ dx = \sqrt{2}\operatorname{sh}t\,dt \\ \frac{x}{\sqrt{2}} = \operatorname{ch}t \\ t = \operatorname{arch}\frac{x}{\sqrt{2}} \end{array}\right| = \int \underbrace{\underbrace{\sqrt{2\operatorname{ch}^2 t - 2}}_{2\operatorname{sh}^2 t}}_{\sqrt{2}\operatorname{sh}t}\sqrt{2}\operatorname{sh}t\,dt =$$

$$= \int 2\operatorname{sh}^2 t\,dt = 2\int\frac{\operatorname{ch}2t - 1}{2}\,dt = \frac{\operatorname{sh}2t}{2} - t + C =$$

$$\frac{\operatorname{sh}2t}{2} = \frac{\cancel{2}\operatorname{sh}t\operatorname{ch}t}{\cancel{2}} = \sqrt{\operatorname{ch}^2 t - 1}\,\operatorname{ch}t = \sqrt{\left(\frac{x}{\sqrt{2}}\right)^2 - 1}\;\frac{x}{\sqrt{2}}$$

$$(*) = \sqrt{\left(\frac{x}{\sqrt{2}}\right)^2 - 1}\;\frac{x}{\sqrt{2}} - \operatorname{arch}\frac{x}{\sqrt{2}} + C$$

$$\int\sqrt{1-\underbrace{9x^2}_{(3x)^2}}\,dx = \left|\begin{array}{l} 3x = \sin t \\ 3\,dx = \cos t\,dt \\ dx = \frac{1}{3}\cos t\,dt \\ t = \arcsin 3x \end{array}\right| = \int \underbrace{\underbrace{\sqrt{1-\sin^2 t}}_{\cos^2 t}}_{\cos t}\frac{1}{3}\cos t\,dt =$$

$$= \frac{1}{3}\int\cos^2 t\,dt = \frac{1}{3}\int\frac{1+\cos 2t}{2}\,dt = \frac{1}{6}\left[t + \frac{\sin 2t}{2}\right] + C =$$

$$= \frac{1}{6}\left[\arcsin 3x + 3x\sqrt{1-9x^2}\right] + C$$

$$\frac{\sin 2t}{2} = \frac{\cancel{2}\sin t \cos t}{\cancel{2}} = \sin t \sqrt{1-\sin^2 t} = 3x\sqrt{1-(3x)^2}$$

$$\int x\sqrt{1-x^2}\,dx = \left| \begin{array}{l} x = \sin t \\ dx = \cos t\,dt \end{array} \right| = \int \sin t \underbrace{\sqrt{\underbrace{1-\sin^2 t}_{\cos^2 t}}}_{\cos t} \cos t\,dt =$$

$$= -\int \underbrace{-\sin t}_{f'} \underbrace{\cos^2 t}_{f^n}\,dt = -\frac{\cos^3 t}{3} + C = -\frac{\left(\sqrt{1-\sin^2 t}\right)^3}{3} + C =$$

$f = \cos t \quad n = 2$

$f' = -\sin t$

③ $$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$$

$(n \neq -1)$

$$= -\frac{\left(\sqrt{1-x^2}\right)^3}{3} + C = -\frac{1}{3}(1-x^2)^{3/2} + C$$

$$\int x\sqrt{1-x^2}\,dx = \int x(1-x^2)^{1/2}\,dx = -\frac{1}{2}\int \underbrace{-2x}_{f'} \underbrace{(1-x^2)^{1/2}}_{f^n}\,dx =$$

$f = 1-x^2 \quad n = 1/2$

$f' = -2x$

$$= -\frac{1}{2}\,\frac{(1-x^2)^{3/2}}{3/2} + C = -\frac{1}{3}(1-x^2)^{3/2} + C$$

$$\int x\sqrt{1-x^2}\,dx = \int x(1-x^2)^{1/2}\,dx = \left| \begin{matrix} 1-x^2=t \\ -2x\,dx = dt \\ x\,dx = -\frac{1}{2}dt \end{matrix} \right| = \int t^{\frac{1}{2}}\left(-\frac{1}{2}dt\right) =$$

$$(1-x^2)' = -2x$$

$$= -\frac{1}{2}\frac{t^{3/2}}{3/2} + C = -\frac{1}{3}t^{3/2} + C = \underline{\underline{-\frac{1}{3}(1-x^2)^{3/2} + C}}$$

$$\int \underbrace{\sin x}_{f}\,\underbrace{\cos x}_{f'}\,dx = \underline{\underline{\frac{\sin^2 x}{2} + C_1}}\,;$$

$$-\int \underbrace{\sin x}_{f'}\,\underbrace{\cos x}_{f}\,dx = \underline{\underline{-\frac{\cos^2 x}{2} + C_2}}$$

$$\int \underbrace{\sin x\cos x}_{\frac{1}{2}\sin 2x}\,dx = \frac{1}{2}\int \sin 2x\,dx = \underline{\underline{\frac{1}{2}\,\frac{-\cos 2x}{2} + C_3}}$$

---

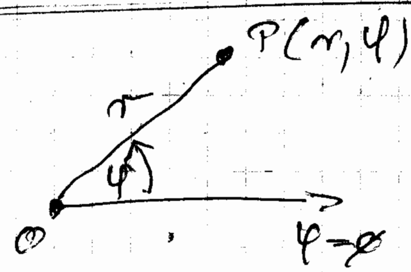
e^x racionális törtfüggvények integrálása

$$\int R(e^x)\,dx = \left| \begin{matrix} e^x = t \\ e^x dx = dt \\ dx = \frac{1}{t}dt \end{matrix} \right| = \int R(t)\,\frac{1}{t}\,dt = \ldots$$

$$\int \frac{e^{2x}}{e^x+1}\,dx = \left| \begin{matrix} e^x = t \\ e^x dx = dt \\ dx = \frac{1}{t}dt \end{matrix} \right| = \int \frac{t^2}{t+1}\,\frac{1}{t}\,dt = \underbrace{\int \frac{t+1-1}{t+1}\,dt}_{\text{áltört}} =$$

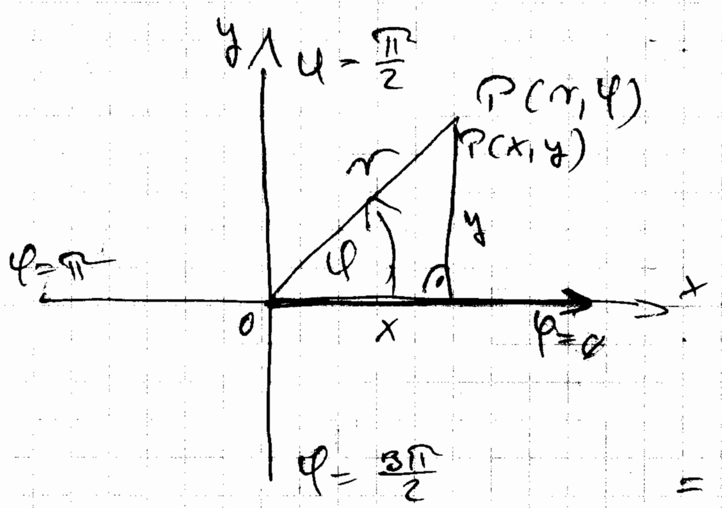
$$= \int \left(1 - \frac{1}{t+1}\right)dt = t - \ln|t+1| + C = \underline{\underline{e^x - \ln(e^x+1) + C}}$$

Polárkoordináta rendszer



$\boxed{r \geq 0; \quad 0 \leq \varphi < 2\pi}$

Kapcsolat a Polár koordrendsz és a Descartes-féle koord. rend. változói között.



$$\cos\varphi = \frac{x}{r} \Rightarrow x = r\cdot\cos\varphi$$

$$\sin\varphi = \frac{y}{r} \Rightarrow y = r\cdot\sin\varphi$$

$$x^2+y^2 = r^2\cdot\cos^2\varphi + r^2\cdot\sin^2\varphi =$$

$$= r^2\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1}$$

$$\boxed{x^2+y^2=r^2}$$

$$\frac{y}{x} = \frac{\sin\varphi}{\cos\varphi} = \operatorname{tg}\varphi$$

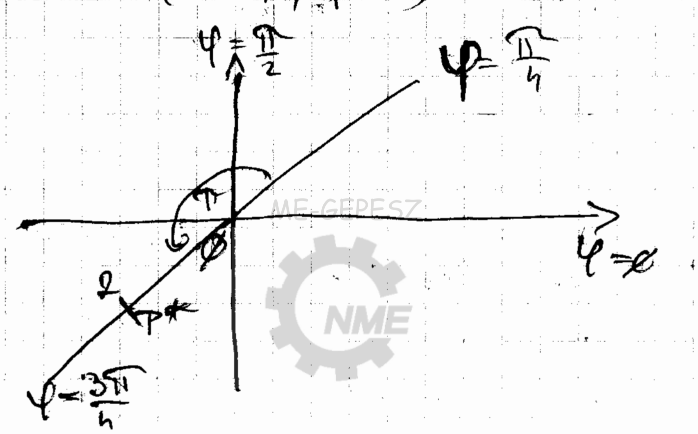
$$\boxed{\varphi = \operatorname{arctg}\frac{y}{x}}$$

Megállapodás szerint:

$$P(\overset{<0}{-r}, \varphi) \to P^*(|-r|; \varphi+\pi)$$

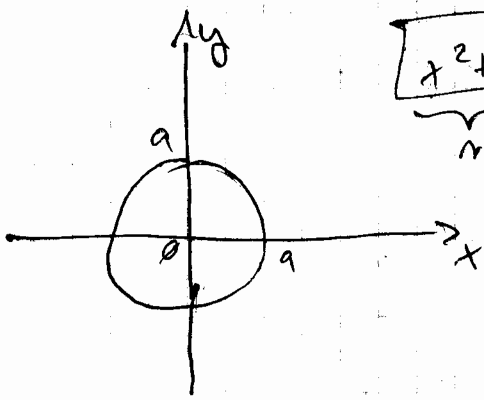
$P(-2; \frac{\pi}{4})$

$P^*(2; \frac{5\pi}{4})$



~~Görbék egyenlete polárkoordináta rendszerben~~

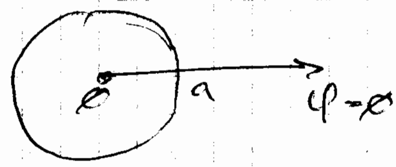
1, Kör egyenlete

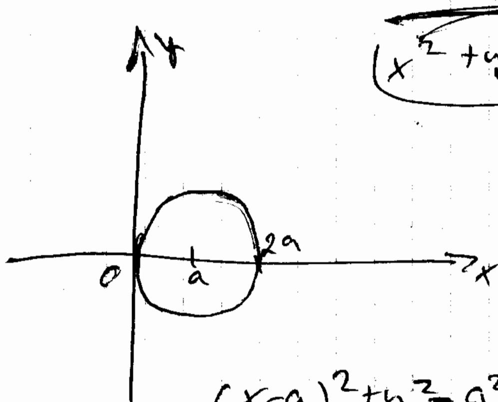


$$\boxed{x^2+y^2=a^2}$$

$$\underbrace{x^2+y^2}_{r^2}: \quad r^2=a^2 \Rightarrow \boxed{r=a}$$

$$\boxed{r=a}$$



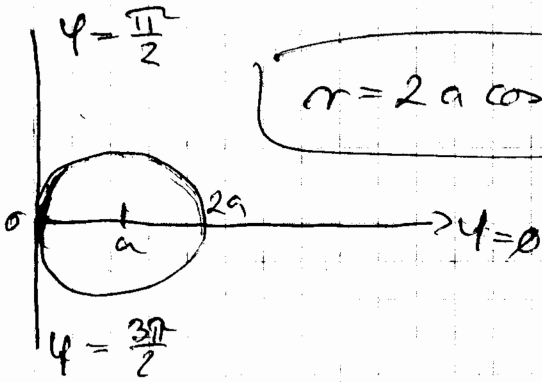


$$\boxed{x^2+y^2=2ax}$$

$$(x-a)^2+y^2=a^2$$

$$x^2-2ax+a^2+y^2=a^2$$

$$x^2-2ax+y=0$$



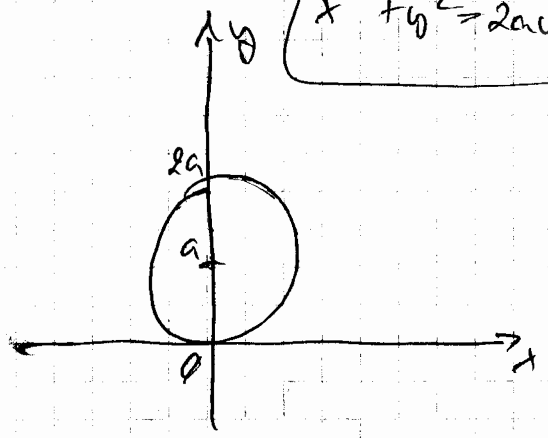
$$\boxed{r=2a\cos\varphi}$$

$$\underbrace{x^2+y^2}_{r^2}=2ax$$

$$r^2 = 2ar\cdot\cos\varphi$$

$$r^2-2ar\cos\varphi=0$$

$$r(r-2a\cos\varphi=0$$

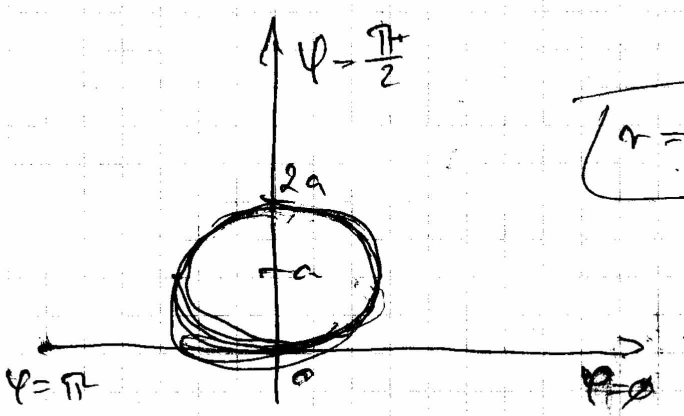


$$\boxed{x^2+y^2=2ay}$$

$$\boxed{x^2+(y-a)^2=a^2}$$

$$x^2+y^2-2ay+a^2=a^2$$

$$x^2+y^2-2ay=0$$



$$\boxed{r=2a\sin\varphi}$$

$$\underbrace{x^2+y^2}=2ay$$

$$r^2=2ar\sin\varphi$$

$$r^2-2ar\sin\varphi=0$$

$$r(r-2a\sin\varphi)=0$$

2, Egyenes egyenlete

Ált. alak: $Ax+By-C=0$

ahol az A és a B egyidejűleg nem lehet nulla

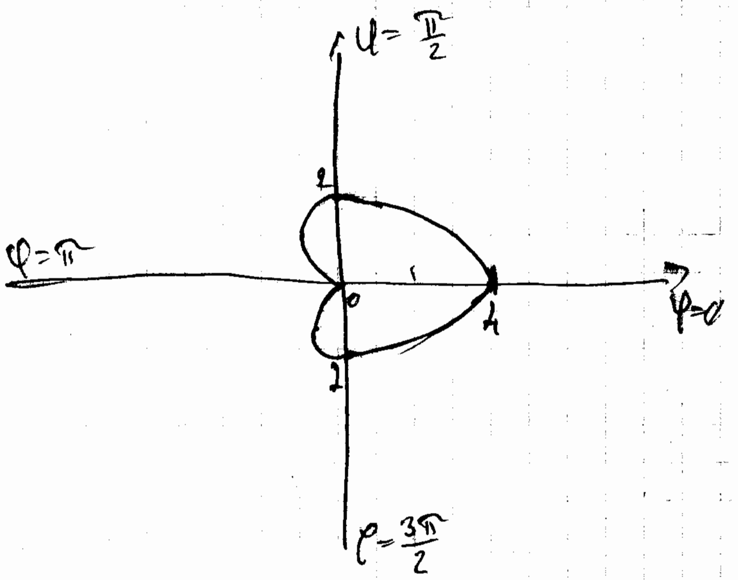
polárban: $A r\cos\varphi+B r\sin\varphi-C=0$

ME-GEPESZ
NME

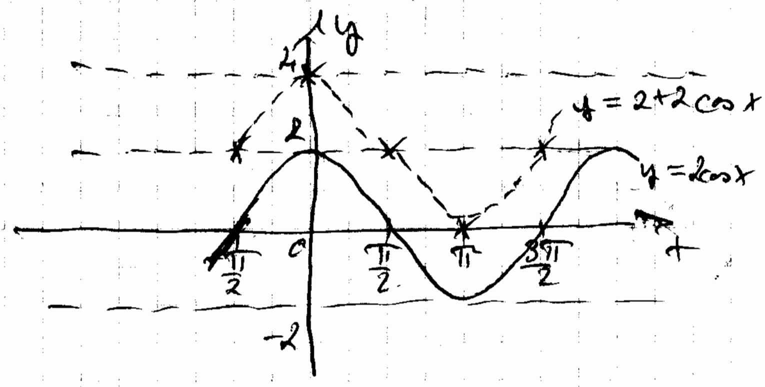
3; Kardioid (Szívgörbe)

$$r = a(1 \pm \cos\varphi)$$
$$r = a(1 \pm \sin\varphi)$$

$r = 2+2\cos\varphi$



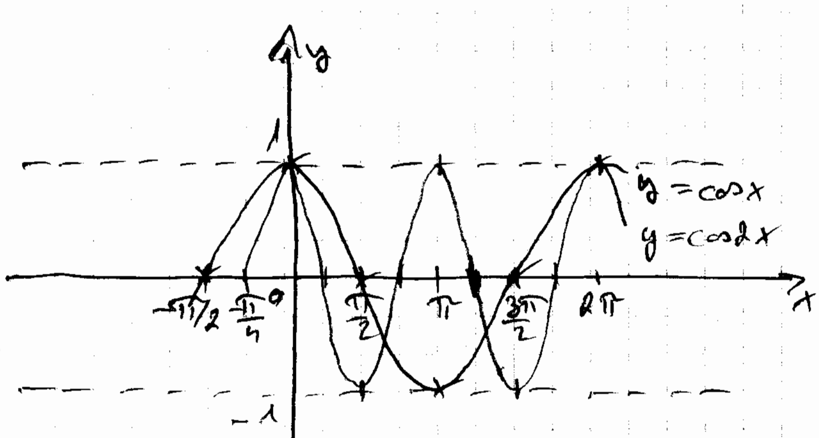
~~y = 2~~ $y = 2+2\cdot\cos x$

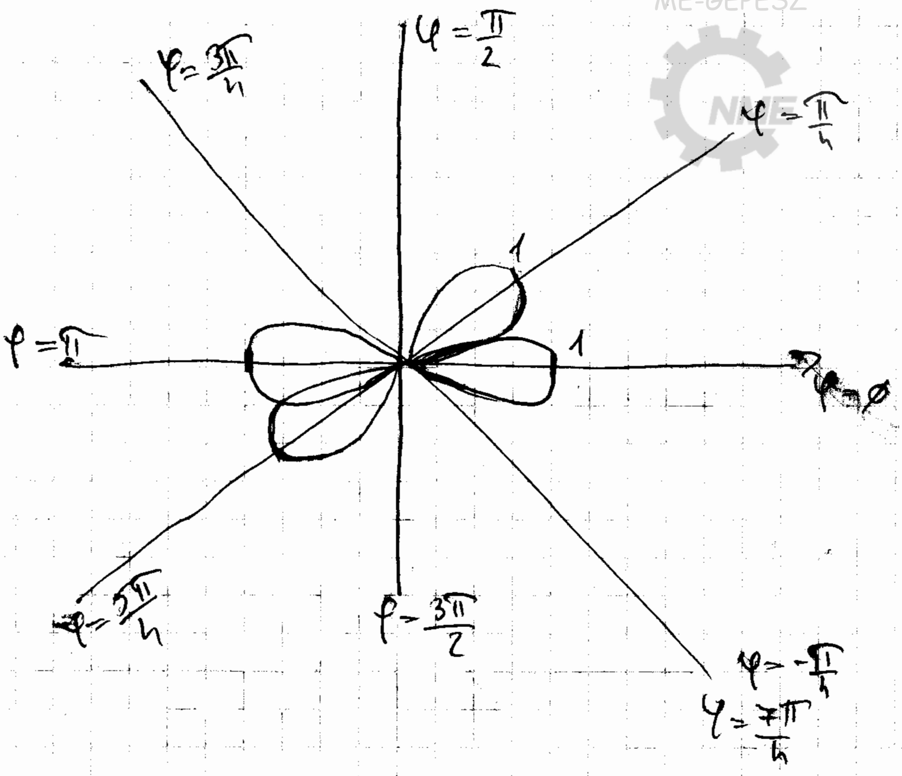


Hf.:
$r = 1-\cos\varphi$
$r = 1+\sin\varphi$
$r = 1-\sin\varphi$

4. Lemniszkáta

$\boxed{r = a\sqrt{\cos 2\varphi}}$ $\boxed{r = a\sqrt{\sin 2\varphi}}$





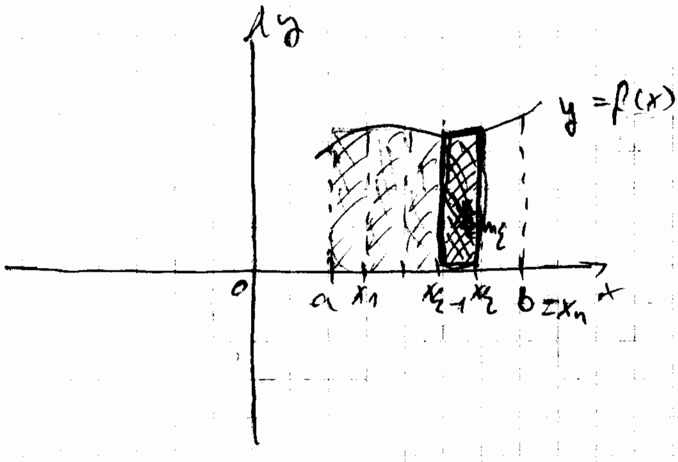
Hf.
$r = \sqrt{\sin 2\varphi}$

5, Spirálisok (lásd. Mat. I.)

r mattrpme

Határozott integrál

Tekintsük az ab zárt intervallumon értelmezett folytonos f fgv.-t (=> korlátos)



1. lépés. Beosztjuk az intervallumot n egyenlő részre.

$$a = x_0 < x_1 < x_2 < \dots < x_{k-1} < x_k < \dots = b = x_n$$

2., jelölés $\Delta x_k = x_k - x_{k-1}$

$$s_n = \sum_{k=1}^{n} m_k \Delta x_k$$ alsó összeg

$m_k = \min f(x)$ $[x_{k-1}, x_k)$

$M_k = \max f(x)$, —"—

$$S_k = \sum_{k=1}^{n} M_k \Delta x_k$$ felső összeg

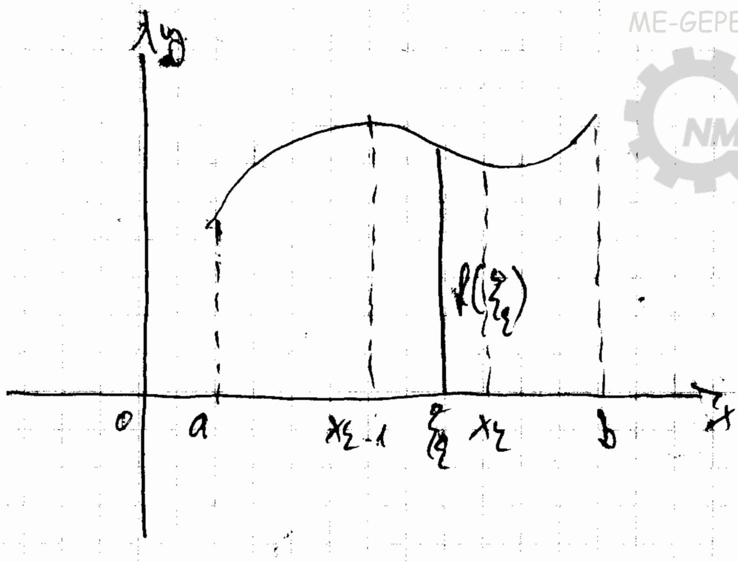
$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} S_n = \int_a^b f(x)\,dx$$

$\max \Delta x_k \to 0$ $\max \Delta x_k = 0$

Ezt a közös határértéket a f fgv. ab zárt intervallumon vett határozott integráljának, vagy Riemann integráljának nevezzük.

a - alsó határ

b - felső —"—



$$\sum_{k=1}^{n} f(\xi_k)\,\Delta x_k$$

integrál közelítő összeg

$$S_n = \int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}$$

Ha az f fgv-nek valamely intervallumon vett határozott integrálja létezik, akkor a fgv. az intervallumon Riemann szerint integrálható.

Ha az f fgv. valamely zárt intervallumon folytonos akkor ott integrálható.

Határozott integrál kiszámítása

Tulajdonságok: tételezzük fel, hogy az f és a g fgv integrálhatóak ezen az intervallumon.

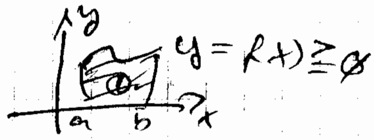
(1) $$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

(2) $$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

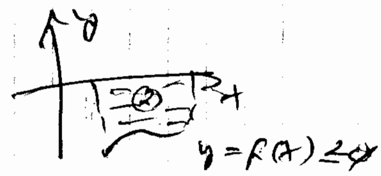
(3) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

a c b

(4) Ha $f(x) \geq 0$, akkor $\int_a^b f(x)\,dx \geq 0$

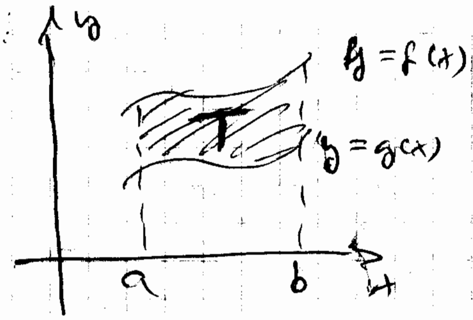


Ha $f(x) \leq 0$, akkor $\int_a^b f(x)\,dx \leq 0$



Következménye: Ha $f(x) \geq g(x)$, akkor

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$



$$\underbrace{\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0}$$

$$\int_a^b [f(x)-g(x)]\,dx \geq 0 \Rightarrow \boxed{T = \int_a^b [f(x)-g(x)]\,dx}$$

---

(5) $$\int_a^b f(x)\,dx = -\int_a^b f(x)\,dx \Rightarrow \int_a^a f(x)\,dx = 0$$

Newton-Leibniz szabály (formula)

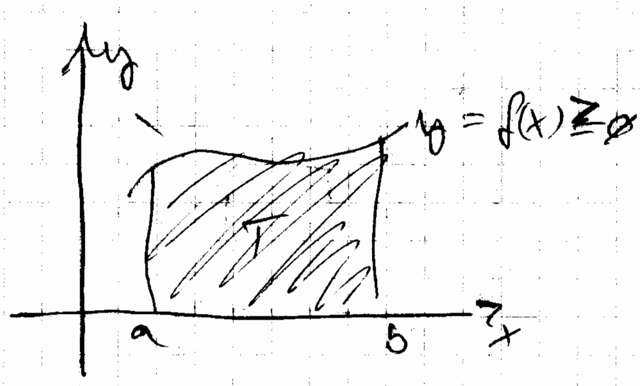
Ha az f fgv integrálható a zárt ab intervallumon és F a f fgv primitív fgv-e akkor

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

ahol $F'(x) = f(x)$

A határozott integrál geometriai jelentése

Ha az $[a,b]$ zárt int-on $f(x) \geq 0$ akkor a határozott integrál a görbe az $[a,b]$ zárt int. az $x=a$ és $x=b$ egyenes által közrezárt terület mérőszáma

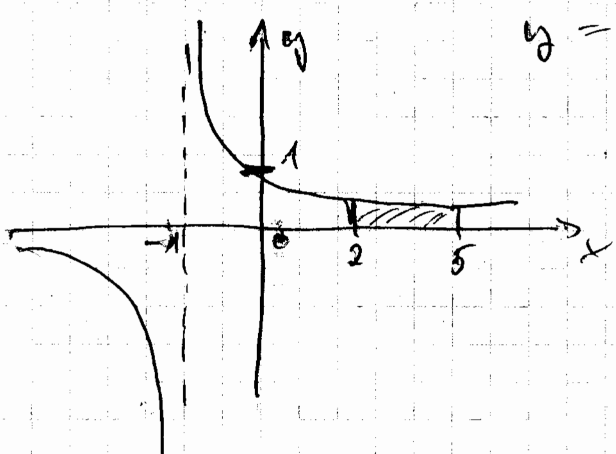


$$T = \int_{x=a}^{b} f(x)\,dx$$

$$\int_2^5 \frac{1}{x+1}\,dx = \Big[\ln|x+1|\Big]_2^5 = \ln 6 - \ln 3 = \ln 2$$

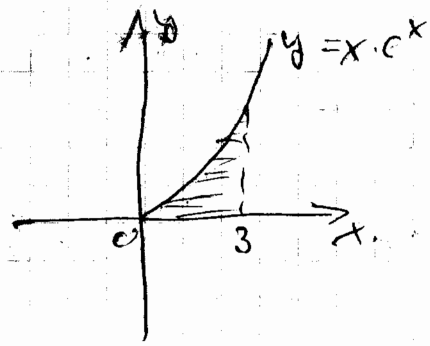
terület egység

$$y = \frac{1}{x+1}$$



Parciális integrálás

$$\int_0^3 x e^x\,dx = \begin{vmatrix} u = x & v' = e^x \\ u' = 1 & v = e^x \end{vmatrix} =$$

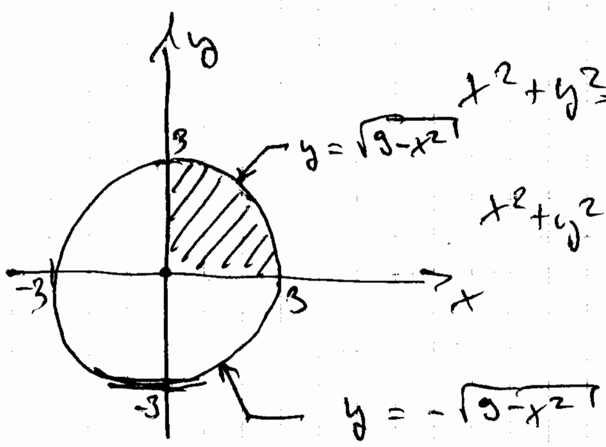


$$= \Big[xe^x\Big]_0^3 - \int_0^3 e^x\,dx = \Big[xe^x - e^x\Big]_0^3 = 3e^3 - e^3 - (0-1) = \underline{\underline{1+2e^3}} \text{ t.e}$$

terület egység

Helyettesítéses integrálás

Számítsa ki integrál számítás segítségével a 3 sugarú kör területét.



$$x^2+y^2=3^2$$

$$x^2+y^2=9 \Rightarrow y^2=9-x^2$$

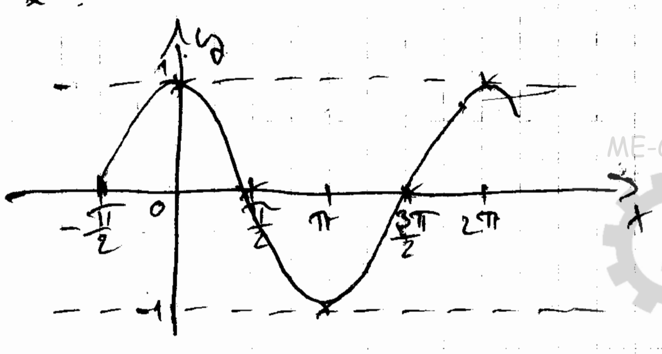
$$y=\pm\sqrt{9-x^2}$$

$$\int_{x=0}^{3}\sqrt{9-x^2}\,dx=\left|\begin{array}{l} x=3\sin t \\ dx=3\cos t\,dt \\ \begin{array}{c|c|c} x & 0 & 3 \\ \hline t & 0 & \frac{\pi}{2}\end{array}\end{array}\right|=\int_{t=0}^{\pi/2}\underbrace{\sqrt{\underbrace{9-9\sin^2 t}_{9\cos^2 t}}}_{3\cos t}\;3\cos t\,dt=$$

$$=\int_{t=0}^{\pi/2}9\underbrace{\cos^2 t}_{\frac{1+\cos 2t}{2}}\,dt=\frac{9}{2}\left[t+\frac{\sin 2t}{2}\right]_0^{\pi/2}=\frac{9}{2}\left[\frac{\pi}{2}+\frac{\sin\pi}{2}-0\right]=\frac{9}{4}\pi$$

$$T_{kör}=4\cdot\frac{9}{4}\pi=9\pi\ \text{(ter. e.)}$$

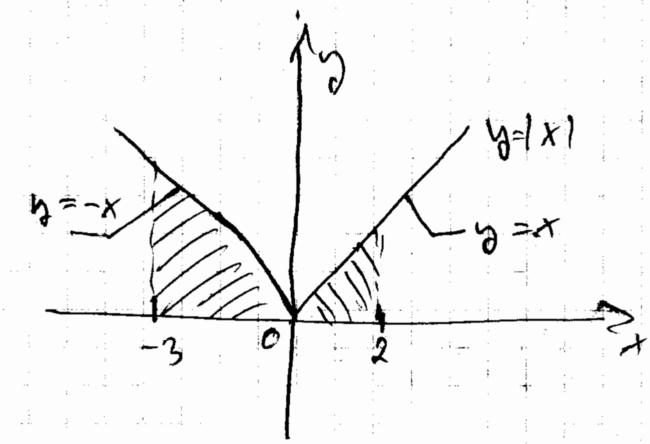
$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos x\,dx=\left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}=\underbrace{\sin\frac{3\pi}{2}}_{-1}-\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1}=-1+1=0$$



$$\int_{-3}^{2}|x|\,dx = \int_{-3}^{0}-x\,dx + \int_{0}^{2}x\,dx = \left[-\frac{x^2}{2}\right]_{-3}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2} =$$

$$= 0-\left(-\frac{(-3)^2}{2}\right)+\frac{2^2}{2}-0 = \frac{9}{2}+2=\frac{13}{2}$$

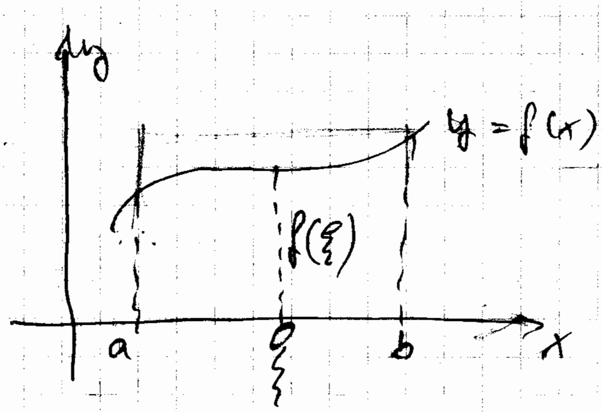
t.e.



Integrál számítás középérték tétele

<u>Tétel:</u> ha a f fgv. folytonos a [a,b] intervallumon akkor van egy olyan ξ hely ahol

$$\boxed{f(\xi)=\frac{1}{b-a}\int_a^b f(x)\,dx}$$

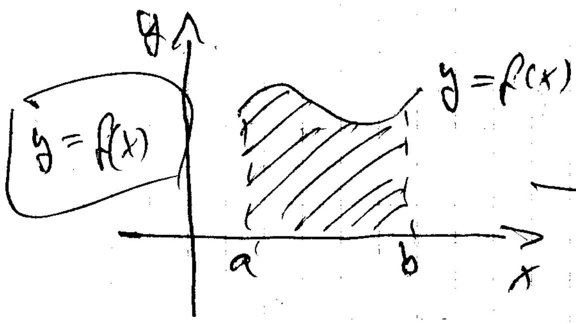


$$f(\xi)(b-a)=\int_a^b f(x)\,dx$$

az $f(\xi)$ értéke az f fgv. a;b intervallumon vett integrál középértékének nevezzük (ez az $f(x)$ fgv. értékei átlaga a számtani közép általánosítása)

Határozott integrál alkalmazásai

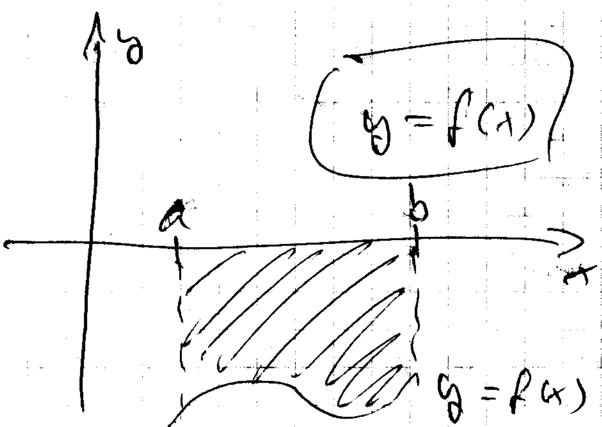
Területszámítás



f folytonos (=> korlátos) függv.

→ görbe vonalú trapéz

$$\boxed{T := \int_a^b f(x)\,dx}\text{, ha } f(x) \geq 0$$



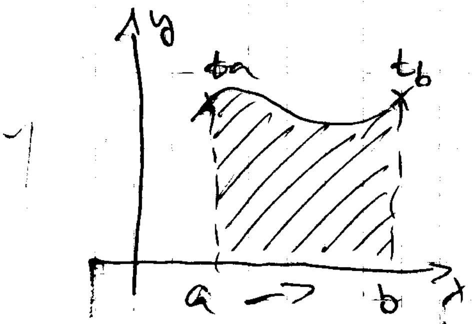
$$T = \left| \int_a^b f(x)\,dx \right|\text{, ha } f(x) \leq 0$$

Ha a görbe egyenlete:

$$\left.\begin{array}{l} x = x(t) \\ y = y(t) \end{array}\right\}$$

$$\dot{x}(t) = \frac{dx}{dt} \Rightarrow$$

$$\Rightarrow dx = \dot{x}(t)\,dt$$



$$\int_{x=a}^{b} f(x)\,dx = \int_{x=a}^{b} y(x)\,dx = \int_{t=t_a}^{t_b} y(t)\,\dot{x}(t)\,dt$$

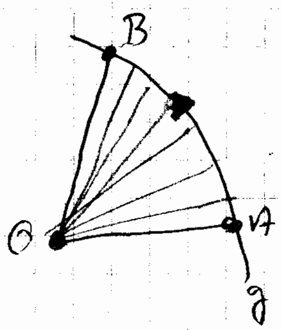
összefoglalva:

$$\left.\begin{array}{l} x = x(t) \\ y = y(t) \end{array}\right\}$$

$$\boxed{T = \int_{t=t_a}^{t_b} y(t)\,\dot{x}(t)\,dt}$$

Szektorterület

Szektor : fontos síkidom



$$g:\ \left.\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}\right\}\quad t_a \le t \le t_b$$

$$\boxed{S := \frac{1}{2}\int_{t=t_a}^{t_b} \left[x(t)\dot{y}(t) - \dot{x}(t)y(t)\right] dt}$$

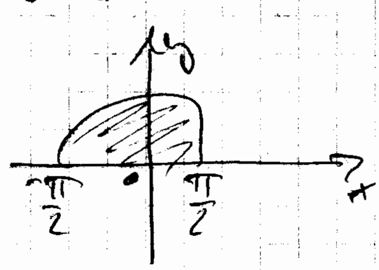
$g:\ r = r(\varphi) \qquad \varphi_A \le \varphi \le \varphi_B$

$$\boxed{S := \frac{1}{2}\int_{\varphi=\varphi_A}^{\varphi_B} r^2(\varphi)\, d\varphi}$$

Ha a görbén A-ból haladok B felé akkor a S pozitív forditott irányban való haladás esetén S negatív.

Példák:

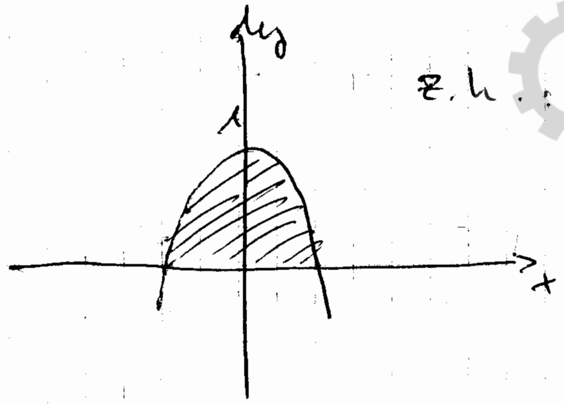
Számolják ki az $y = \cos x$ görbe egy ívének az x tengellyel bezárt területét



$$T = \int_{-\frac{\pi}{2}}^{\pi/2} \cos x\, dx = \left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \underbrace{\sin\frac{\pi}{2}}_{1} - \underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1} = 2 \ (\text{ter. e.})$$

Határozzuk meg az $y=1-x^2$ görbe és



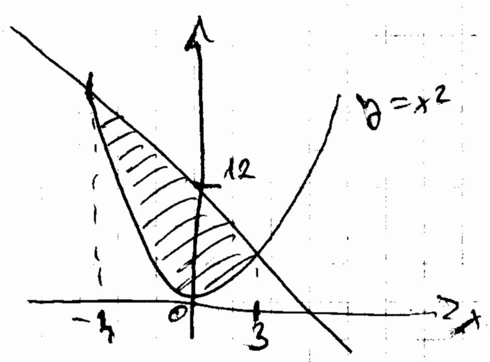
z.h.: $1-x^2=0$

$x_{1,2}=\pm 1$

$$T=\int_{-1}^{1}(1-x^2)\,dx=\left[x-\frac{x^3}{3}\right]_{-1}^{1}=$$

$$=2\cdot\left[x-\frac{x^3}{3}\right]_{0}^{1}=2\left[1-\frac{1}{3}-0\right]=\frac{4}{3}\ (\text{ter. e.})$$

$$\left.\begin{aligned} y&=x^2 \\ y&=-x+12 \end{aligned}\right\}$$



metszéspont:

$$\left.\begin{aligned} y&=x^2 \\ y&=-x+12 \end{aligned}\right\}$$

$x^2=-x+12$

$x^2+x-12=0$

$x_{1,2}=\begin{cases}-4\\ 3\end{cases}$

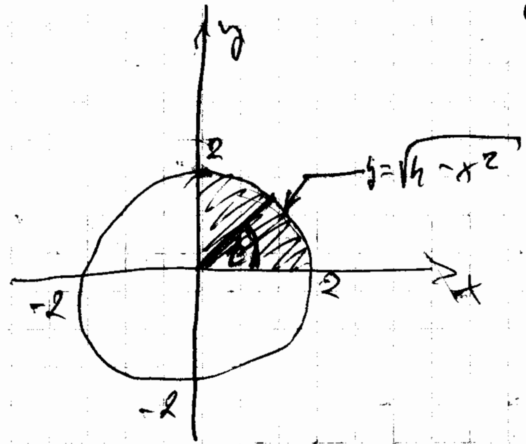
Ha $f(x)\geq g(x)$, akkor $\boxed{T=\int_a^b\left[f(x)-g(x)\right]dx}$

$$T=\int_{-4}^{3}\left[-x+12-x^2\right]dx=\left[-\frac{x^2}{2}+12x-\frac{x^3}{3}\right]_{-4}^{3}=$$

$$=-\frac{9}{2}+36-\frac{27}{3}-\left(-\frac{(-4)^2}{2}+12(-4)-\frac{(-4)^3}{3}\right)=-\frac{9}{2}+27+8+48-\frac{64}{3}=$$

$= \ldots$

Számítsa ki az $r=2$ sugarú kör területét



a, $x^2+y^2=4 \Rightarrow y=\pm\sqrt{4-x^2} \Rightarrow y=\sqrt{4-x^2}$

$$T = 4\int_{x=0}^{2}\sqrt{4-x^2}\,dx = \left|\begin{array}{l} x=2\sin t \\ dx=2\cos t\,dt \\ \begin{array}{c|c|c} x & 0 & 2 \\ \hline t & 0 & \frac{\pi}{2} \end{array}\end{array}\right| =$$

$$= 4\cdot\int_{t=0}^{\pi/2}\underbrace{\sqrt{4-4\cdot\sin^2 t}}_{2\cos t}\;2\cdot\cos t\,dt = 4\int_{t=0}^{\pi/2}4\cdot\cos^2 t\,dt =$$

$$= 16\int_{t=0}^{\pi/2}\frac{1+\cos 2t}{2}\,dt = 8\int_{t=0}^{\pi/4}[1+\cos 2t]\,dt =$$

$$= 8\left[t+\frac{\sin 2t}{2}\right]_0^{\pi/2} = 8\cdot\left[\frac{\pi}{2}+\frac{\sin\pi}{2}-0\right] = 4\pi \quad (\text{t.e.})$$

Paraméteresen

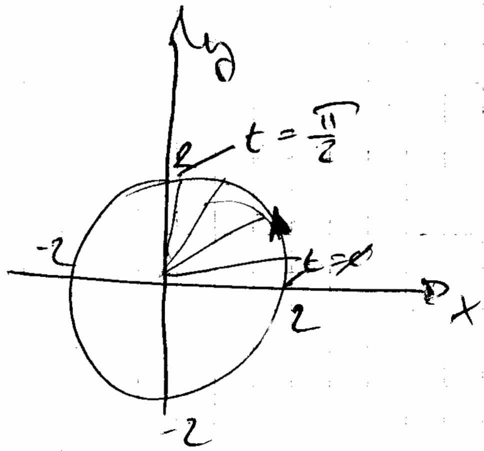
b, $\left.\begin{array}{l} x = 2\cos t \\ y = 2\sin t \end{array}\right\}$ $(0\le t\le 2\pi)$

sugár

$$\int_{x=a}^{b} y(x)\,dx \rightarrow T=\int_{t=t_a}^{t_b} y(t)\,\dot{x}(t)\,dt$$

$$T = 4\int_{t=\frac{\pi}{2}}^{0} 2\sin t\,(-2\sin t)\,dt = 16\int_{t=\frac{\pi}{2}}^{0} -\sin^2 t\,dt = 16\int_{t=0}^{\pi/2}\frac{1-\cos 2t}{2}\,dt =$$

$$= 8\left[t - \frac{\sin 2t}{2}\right]_0^{\pi/2} = 4\pi \ (t.e.)$$



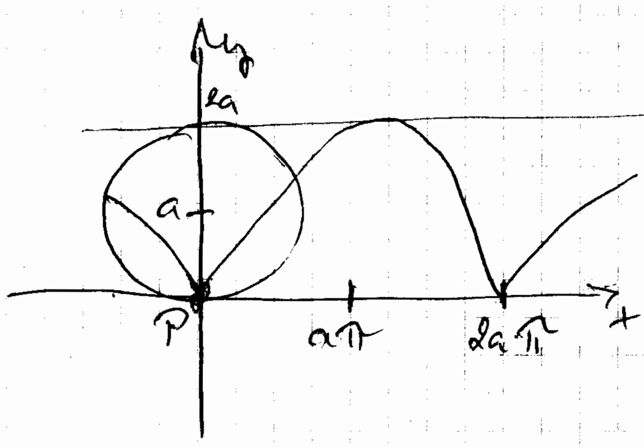
$$S = 4 \cdot \frac{1}{2} \int_{t=0}^{\pi/2} \big(\underbrace{2\cos t \cdot 2\cos t}_{4\cos^2 t} \underbrace{- (-2\sin t) 2\sin t}_{+\,4\sin^2 t}\big)\, dt$$

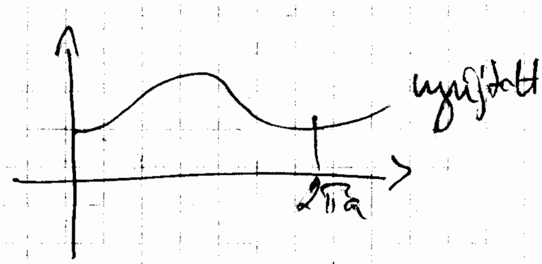
$$= 8 \cdot \int_0^{\pi/2} dt = 8\left[t\right]_0^{\pi/2} = 4\pi \ (t.e.)$$

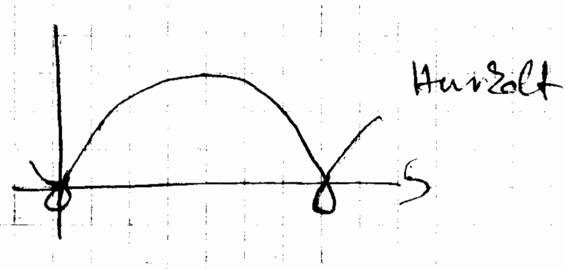
c, Polárban

$\gamma:\ r = 2$

$$S = 4 \cdot \frac{1}{2} \int_{\varphi=0}^{\pi/2} 2^2 \, d\varphi = 8 \int_{\varphi=0}^{\pi/2} d\varphi = 8\left[\varphi\right]_0^{\pi/2} = 4\pi \ (t.e.)$$







Számítsuk ki a ciklois első ~~és az~~ egy íve és az x tengely közötti területét.

$$\left.\begin{array}{l} x = a(t-\sin t) \\ y = a(1-\cos t) \end{array}\right\}$$

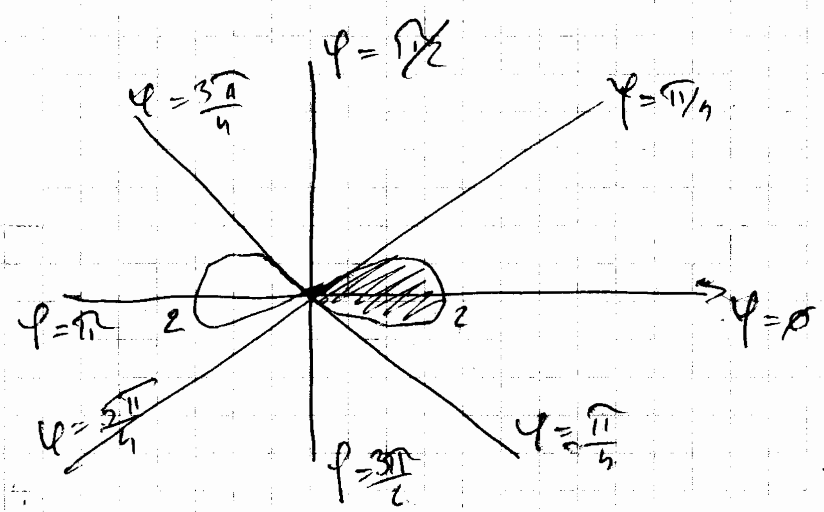
$$T = \int_{t=t_1}^{t_2} y(t)\dot{x}(t)\,dt = \int_{t=0}^{2\pi} \overbrace{a(1-\cos t)\,a(1-\cos t)}^{a^2(1-\cos t)^2}\,dt =$$

$$= a^2\int_{t=0}^{2\pi}(1-2\cos t+\underbrace{\cos^2 t}_{\frac{1+\cos 2t}{2}})\,dt = a^2\left[t-2\sin t+\frac{1}{2}t+\frac{1}{2}\frac{\sin 2t}{2}\right]_0^{2\pi} =$$

$$= a^2\left[2\pi+\frac{1}{2}\cdot 2\pi\right] = 3\pi a^2 \text{ (t.e.)}$$

$$\cancel{\oint} \; S = \frac{1}{2}\int_{t=2\pi}^{0}\left[a(t-\sin t)a\sin t - a(1-\sin t)a(1-\cos t)\right]dt =$$

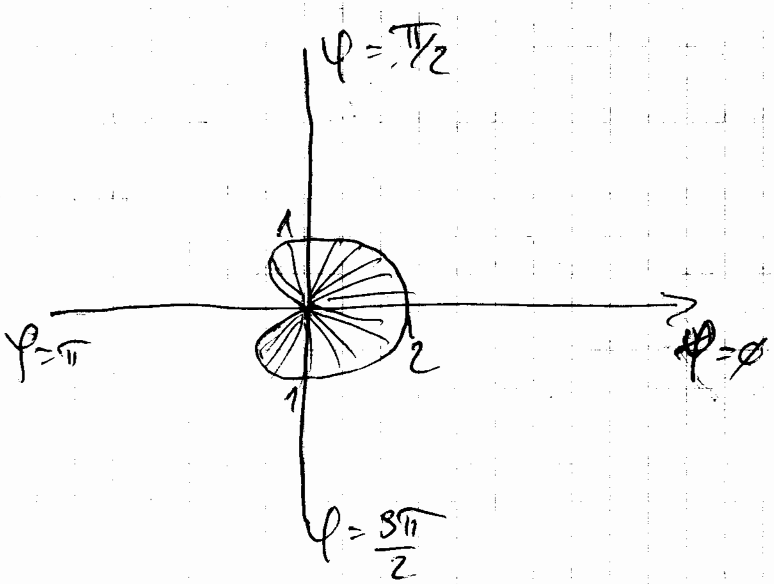
Számoljuk ki $r = 2\sqrt{\cos 2\varphi}$



$$S = \frac{1}{2}\cdot 2\int_{\varphi=-\frac{\pi}{4}}^{\pi/4}\left[2\sqrt{\cos 2\varphi}\right]^2 d\varphi = 2\int_{\varphi=0}^{\pi/4} 4\cdot\cos 2\varphi\,d\varphi =$$

$$= 8\left[\frac{\sin 2\varphi}{2}\right]_0^{\pi/4} = 4\left[\sin\frac{\pi}{2} - 0\right] = 4 \ (t.e.)$$

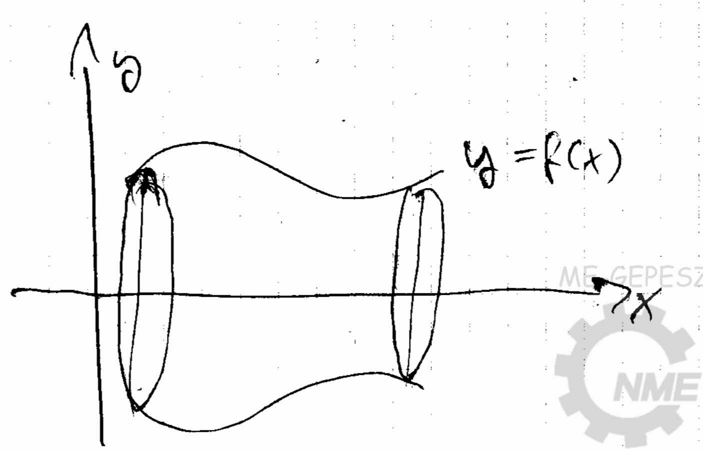
Határozzuk meg az $r = 1 + \cos\varphi$ kardioid területét



$$S = \frac{1}{2}\int_{\varphi=0}^{2\pi}(1+\cos\varphi)^2\, d\varphi = \frac{1}{2}\int_{\varphi=0}^{2\pi}\Big(1 + 2\cdot\cos\varphi + \underbrace{\cos^2\varphi}_{\frac{1+\cos 2\varphi}{2}}\Big)\, d\varphi =$$

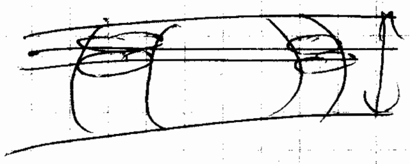
$$= \frac{1}{2}\left[\varphi + 2\sin\varphi + \frac{1}{2}\varphi + \frac{\sin 2\varphi}{4}\right]_0^{2\pi} = \frac{1}{2}\left[2\pi + \frac{1}{2}\cdot 2\pi\right] = \frac{3\pi}{2} \ (te.)$$

Forgástestek térfogata



Cavalieri-elv 8

Ha van két egyenlő magasságú test, és erre a magasságra merőleges síkmetszeti területe megegyezik, akkor a két test térfogata egyenlő



Tétel

A test pontjainak x koordinátái legyenek az $[a;b]$ intervallumban, ha minden x-nél az x tengelyre merőleges sík által a testből kimetszett síkidom területe T_x akkor a test térfogata

$$\boxed{V = \int_{x=a}^{b} T(x)\,dx}$$

$x \in [a;b]$