pót ZH 2007. május 05.

Kettős integrál (feladatok)

Példa

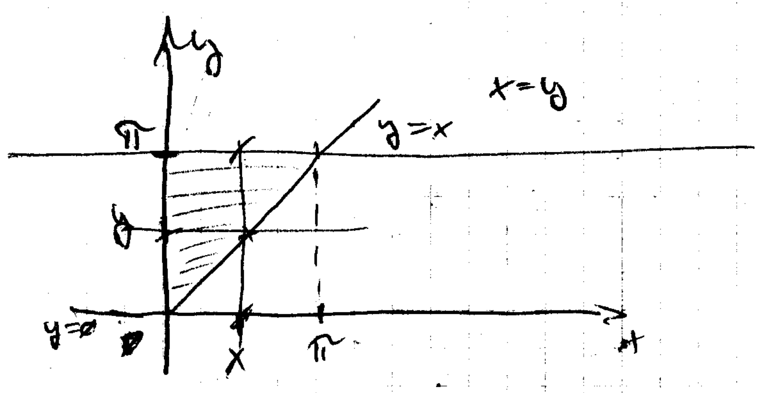
$$\int_{y=0}^{\pi}\int_{x=0}^{y}\cos(x+y)\,dx\,dy = \int_{y=0}^{\pi}\left[\sin(x+y)\right]_{x=0}^{y}dy =$$

$$= \int_{y=0}^{\pi}(\sin 2y - \sin y) =$$

$$= \left[-\frac{\cos 2y}{2} + \cos y\right]_{y=0}^{\pi} =$$

$$= -\frac{\overbrace{\cos 2\pi}^{1}}{2} + \overbrace{\cos\pi}^{-1} - \left(-\frac{\overbrace{\cos 0}^{1}}{2} + \overbrace{\cos 0}^{1}\right) =$$

$$= -\frac{1}{2} - 1 + \frac{1}{2} - 1 = -2$$



geometriai jelentése térfogat

$z = \cos(x+y)$ felület

cserélje fel az integrálás sorrendjét

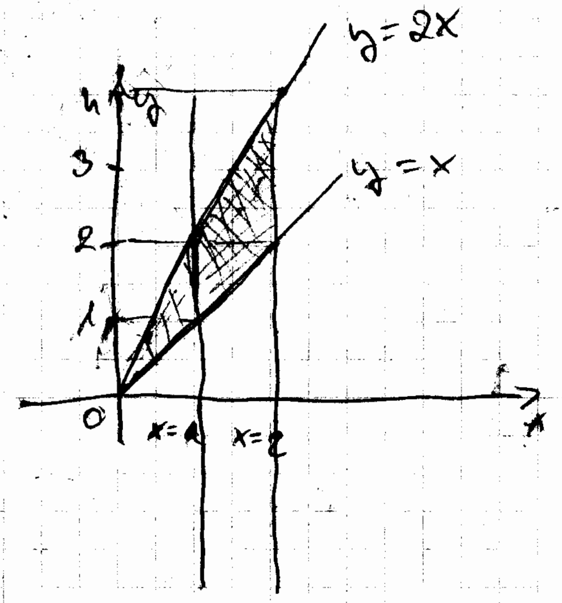
$$\int_{x=0}^{\pi}\int_{y=x}^{\pi}\cos(x+y)\,dx\,dy = \int_{x=0}^{\pi}\left[\sin(x+y)\right]_{y=x}^{\pi}dx =$$

$$= \int_{x=0}^{\pi}\left[\sin(x+\pi) - \sin 2x\right]dx = \left[-\cos(x+\pi) + \frac{\cos 2x}{2}\right]_{x=0}^{\pi} =$$

$$= -\overbrace{\cos 2\pi}^{1} + \frac{\overbrace{\cos 2\pi}^{1}}{2} - \left(-\overbrace{\cos\pi}^{-1} + \frac{\overbrace{\cos 0}^{1}}{2}\right) = -1 + \frac{1}{2} - 1 - \frac{1}{2} = -2$$

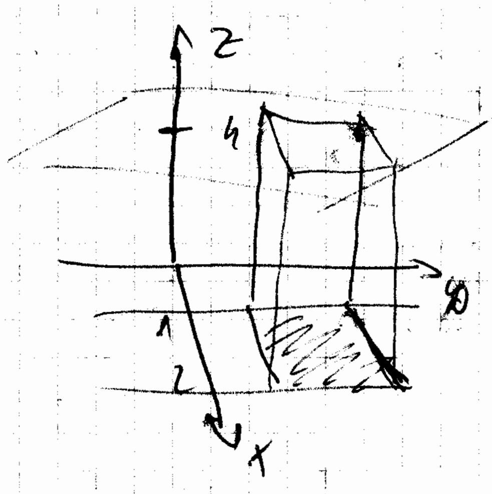
2, Cserélje fel az integrálás sorrendjét a következőkben

$$\int_{x=1}^{2}\int_{y=x}^{2x} f(x,y)\,dx\,dy = \int_{y=1}^{2}\left[\int_{x=1}^{y} f(x,y)\,dx\right]dy + \int_{y=2}^{4}\left[\int_{x=\frac{y}{2}}^{2} f(x,y)\,dx\right]dy$$



$y = 2x \Rightarrow x = \frac{y}{2}$

$z = f(x,y) = 4$ sík



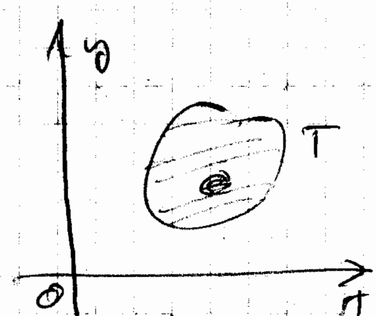
Új változó bevezetése

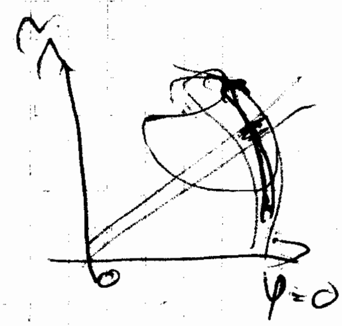
Polárkoordinátákra való áttérés

$(x,y) \to (r,\varphi)$

$$\left.\begin{aligned} x &= r\cdot\cos\varphi \\ y &= r\cdot\sin\varphi \end{aligned}\right\}$$

$dx\,dy = r\,dr\,d\varphi$

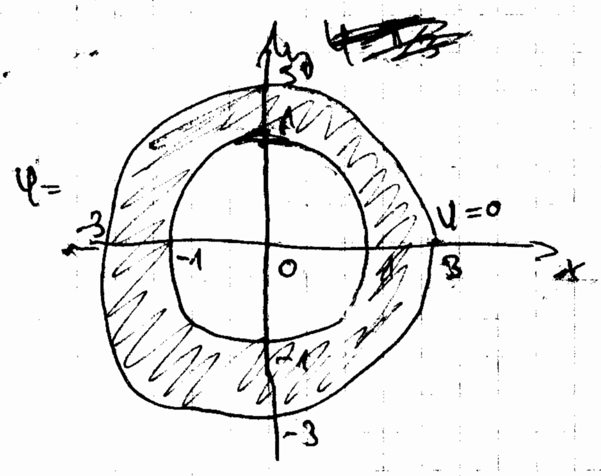




$$\left.\begin{array}{l}-\infty < x < +\infty \\ -\infty < y < +\infty\end{array}\right\} \qquad \begin{array}{l} 0 \le r \le +\infty \\ 0 \le \varphi \le 2\pi \\ (-\pi) \le \varphi \le \pi \end{array}$$

$$\iint_T (1+xy)\,dx\,dy = \left| \begin{array}{l} x = r\cdot\cos\varphi \\ y = r\cdot\sin\varphi \\ dx\,dy = r\,dr\,d\varphi \end{array} \right| = \int_{\varphi=0}^{2\pi}\int_{r=1}^{3} (1 + r\cdot\cos\varphi\, r\cdot\sin\varphi)\, r\,dr\,d\varphi =$$

$T: 1 \le x^2+y^2 \le 9$



$$= \int_{\varphi=0}^{2\pi}\int_{r=1}^{3} [r + r^3 \sin\varphi\cos\varphi]\,dr\,d\varphi =$$

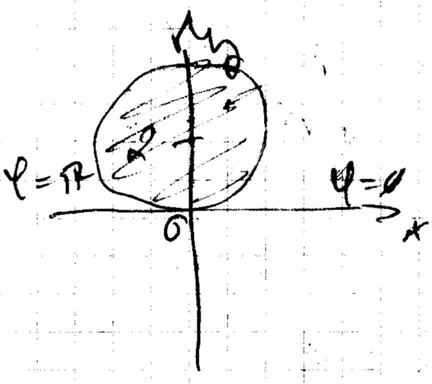
$$= \int_{r=1}^{3} \left[r\varphi + r^3 \frac{\sin^2\varphi}{2} \right]_{\varphi=0}^{2\pi} dr =$$

$$\int \sin x \cos x\,dx = \frac{\sin^2 x}{2} + C$$

$$\int_{r=1}^{3} \left[2\pi r + r^3 \frac{\sin^2 2\pi}{2} - (0+0) \right] dr =$$

$$= \int_{r=1}^{3} 2\pi r\,dr = \pi \int_{r=1}^{3} 2r\,dr = \pi [r^2]^3 = \pi [9-1] = 8\pi$$

$$\iint_T \sqrt{x^2+y^2}\,dx\,dy \qquad T:\ x^2+y^2 \le 4y$$



$x^2+y^2=4y$ ⇒ áttérés polárkoordinátákra

$x^2+y^2-4y=0$

$x^2+(y-2)^2=4$

$r^2=4r\sin\varphi \Rightarrow r^2-4r\sin\varphi=0 \Rightarrow$

$\Rightarrow r(r-4\sin\varphi)=0$

$r=0 \qquad r=4\sin\varphi$

$$\iint_T \sqrt{x^2+y^2}\,dx\,dy = \left| \begin{array}{l} x=r\cdot\cos\varphi \\ y=r\cdot\sin\varphi \\ dx\,dy=r\,dr\,d\varphi \\ x^2+y^2=r^2 \end{array} \right| = \int_{\varphi=0}^{\pi}\int_{r=0}^{4\sin\varphi} \sqrt{r^2}\,r\,dr\,d\varphi =$$

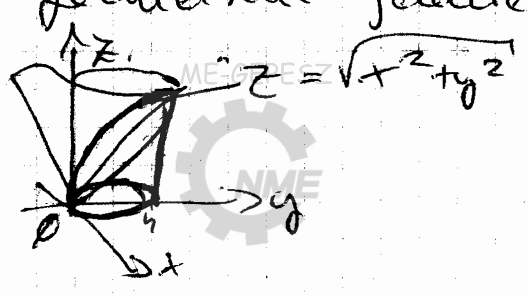
$$\int_{\varphi=0}^{\pi}\int_{r=0}^{4\sin\varphi} r^2\,dr\,d\varphi = \int_{\varphi=0}^{\pi}\left[\frac{r^3}{3}\right]_{r=0}^{4\sin\varphi} d\varphi = \int_{\varphi=0}^{\pi}\left[\frac{1}{3}4^3\sin^3\varphi - 0\right]d\varphi =$$

$$= \frac{4^3}{3}\int_{\varphi=0}^{\pi}\sin\varphi\,(1-\cos^2\varphi)\,d\varphi = \frac{4^3}{3}\int_{\varphi=0}^{\pi}\left[\sin\varphi-\sin\varphi\cos^2\varphi\right]d\varphi =$$

$$= \frac{64}{3}\left[-\cos\varphi+\frac{\cos^3\varphi}{3}\right]_{\varphi=0}^{\pi} = \frac{64}{3}\left[-\cos\pi+\frac{\cos^3\pi}{3} - \left(-\cos 0+\frac{\cos^3 0}{3}\right)\right] = \frac{64}{3}\left[1-\frac{1}{3}+1-\frac{1}{3}\right] = \frac{64}{3}\cdot\frac{4}{3} =$$

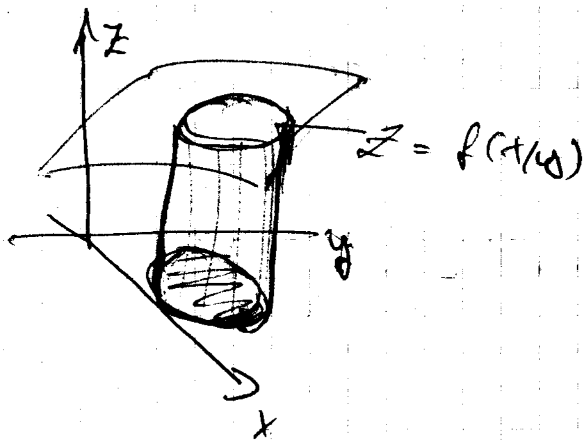
$$= \frac{256}{9}$$

geometriai jelentése: térfogat

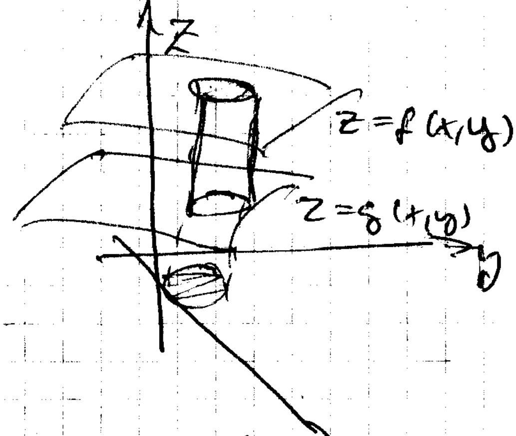


Térfogatszámítás

$$V = \iint_T f(x,y)\,dx\,dy$$



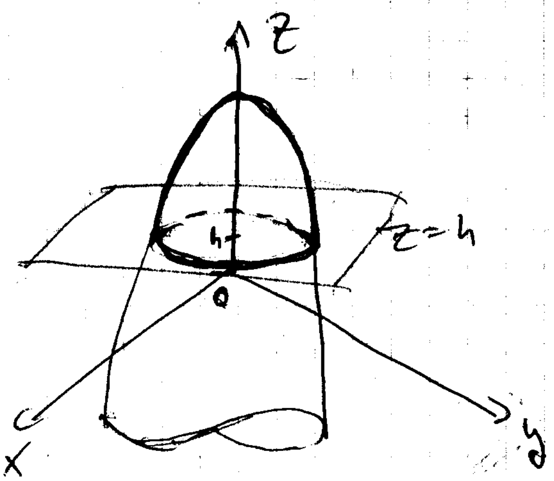
Ha $f(x,y) \geq g(x,y)$ minden $(x,y) \in T$ esetén, akkor



$$V = \iint_T f(x,y)\,dx\,dy - \iint_T g(x,y)\,dx\,dy$$

$$\boxed{V = \iint_T \left[f(x,y) - g(x,y)\right] dx\,dy}$$

$z = 8 - x^2 - y^2 \quad ; \quad z = 4$



$$\left.\begin{aligned} z &= 8 - x^2 - y^2 \\ z &= 4 \end{aligned}\right\}$$

metszésvonala:

$4 = 8 - x^2 - y^2$

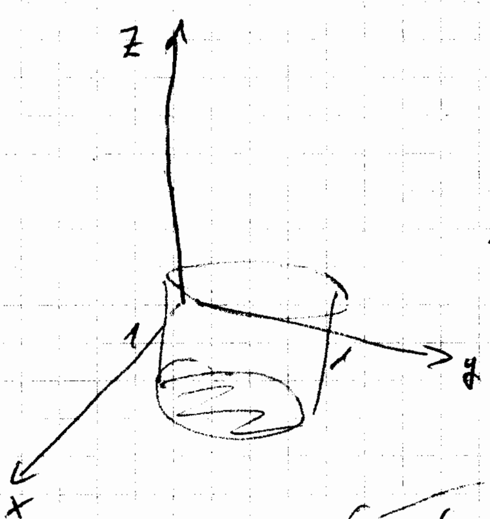
$$\left\{\begin{aligned} x^2 + y^2 &= 4 \\ z &= 4 \end{aligned}\right.$$

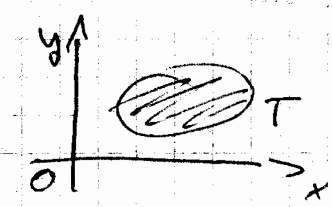
$$V = \iint_T (8-x^2-y^2-4)\,dx\,dy = \iint_T (4-x^2-y^2)\,dx\,dy$$

$$= \left| \begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \\ dx\,dy = r\,dr\,d\varphi \end{array} \right| = \int_{\varphi=0}^{2\pi}\int_{r=0}^{2} (4-r^2)\, r\, dr\, d\varphi =$$

$$= \int_{r=0}^{2} (4r - r^3)\,[\varphi]_0^{2\pi}\, dr = 2\pi \left[2r^2 - \frac{r^4}{4}\right]_0^2 =$$

$$= 2\pi\cdot\left[2\cdot 4 - \frac{16}{4}\right] = \underline{\underline{8\pi}} \quad t.\,e.$$





μT

$$\mu(V) = \mu(T)\cdot 1 = \mu(T)$$

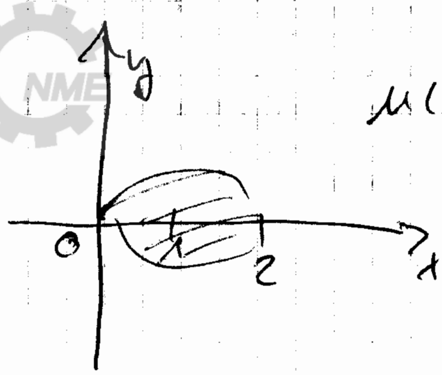
$$\boxed{\mu(T) = \iint_T 1\,dx\,dy = \iint_T dx\,dy}$$

$x^2+y^2 \le 2x$

$x^2+y^2-2x=0$

$x^2-2x+y^2=0$

$(x-1)^2+y^2=1$



$\mu(T)=\iint\limits_T dx\,dy=$

$x^2+y^2=2x \to$ áttérés polárba:

$r^2=2r\cos\varphi$

$r(r-2\cos\varphi)=0$

$r=2\cos\varphi$

$$\mu(T)=\iint\limits_T dx\,dy=\int\limits_{\varphi=\frac{\pi}{2}}^{\pi/2}\int\limits_{r=0}^{2\cos\varphi} r\,dr\,d\varphi=$$

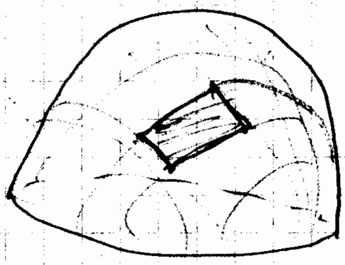
$$=\int\limits_{\varphi=-\frac{\pi}{2}}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2\cos\varphi}d\varphi=\int\limits_{\varphi=-\frac{\pi}{2}}^{\pi/2}2\underbrace{\cos^2\varphi}_{\frac{1+\cos 2\varphi}{2}}d\varphi=\int\limits_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos 2\varphi)\,d\varphi=$$

$$=\left[\varphi+\frac{\sin 2\varphi}{2}\right]_{-\pi/2}^{\pi/2}=\frac{\pi}{2}+\frac{\sin\pi}{2}-\left(-\frac{\pi}{2}+\frac{\sin 2(-\frac{\pi}{2})}{2}\right)=$$

$=\underline{\underline{\pi}}$ (terület egység)

3, felszínszámítás (görbült felület felszíne)

Ha a felület egyenlete $z = f(x,y)$ egyenlettel adott.



$\vec{r} = \vec{r}(u,v)$

$z = f(x,y)$

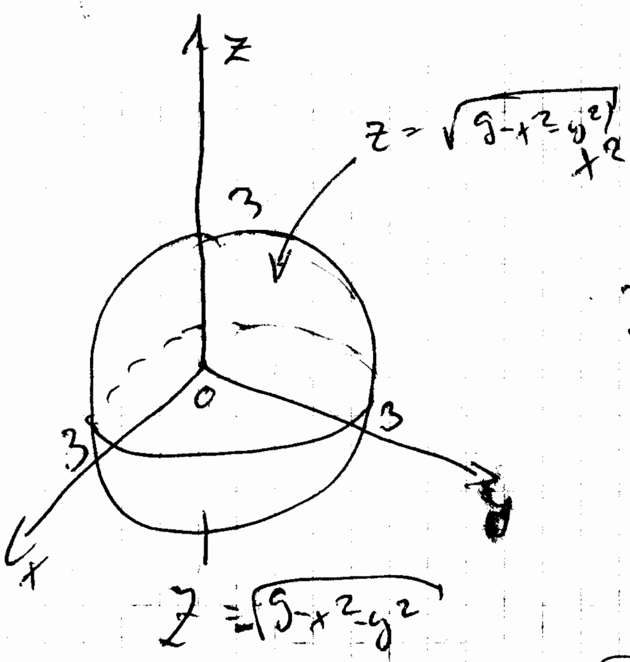
érintővektor: $\vec{n} = (-f'_x, -f'_y, 1)$

$$dF = \sqrt{1 + (f'_x) + (f'_y)^2}\, dx\,dy$$

$$\mu(F) = \iint_T dF \qquad z = f(x,y)$$

$$\mu(F) = \iint_T \sqrt{1 + (f'_x)^2 + (f'_y)^2}\, dx\,dy$$

Számítsuk ki a $r=3$ sugarú gömb térfogatát és felszínét



$$x^2+y^2+z^2=r^2=9 \Rightarrow z=\pm\sqrt{9-x^2-y^2}$$

Térf.: $\mu(V)=$

metszésvonal: $\left.\begin{array}{l} x^2+y^2+z^2=9 \\ z=0 \end{array}\right\}$

$$\left\{\begin{array}{l} x^2+y^2=9 \\ z=0 \end{array}\right. \Rightarrow T: x^2+y^2\le 9$$

$$\mu(V)=2\iint_T \sqrt{9-x^2-y^2}\,dx\,dy = 2\int_{\varphi=0}^{2\pi}\int_{r=0}^{3}\sqrt{9-r^2}\,r\,dr\,d\varphi =$$

$$2\int_{r=0}^{3}(9-r^2)^{\frac{1}{2}}\,r\,[\varphi]_0^{2\pi}\,dr =$$

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}+C$$

$$=4\pi\left[\frac{(9-r^2)^{3/2}}{\frac{3}{2}(-2)}\right]_0^3 = -\frac{4\pi}{3}\left[0-9^{3/2}\right] = \frac{4\pi}{3}\cdot 3^3 = 36\pi$$

térfogat e.

$$dF = \sqrt{1+(f'_x)^2+(f'_y)^2}\,dx\,dy = \sqrt{1+\frac{x^2}{9-x^2-y^2}+\frac{y^2}{9-x^2-y^2}}\,dx\,dy$$

$$f'_x = \frac{1}{2\sqrt{9-x^2-y^2}}\cdot(-2x) = \frac{-x}{\sqrt{9-x^2-y^2}}$$

$$f'_y = \frac{-y}{\sqrt{9-x^2-y^2}}$$

$$\sqrt{\frac{9-x^2-y^2+x^2+y^2}{9-x^2-y^2}}\,dx\,dy = \frac{3}{\sqrt{9-x^2-y^2}}\,dx\,dy$$

félgömb felszíne

$$\mu(F) = \iint_T \frac{3}{\sqrt{9-x^2-y^2}}\,dx\,dy = \int_{\varphi=0}^{2\pi}\int_{r=0}^{3}\frac{3}{\sqrt{9-r^2}}\,r\,dr\,d\varphi =$$

$$= \int_{r=0}^{3} 3\cdot(9-r^2)^{-\frac{1}{2}}\,r\,[\varphi]_0^{2\pi}\,dr = 6\pi\int_{r=0}^{3}(9-r^2)^{-\frac{1}{2}}\,r\,dr =$$

$$= 6\pi\left[\frac{(9-r^2)^{\frac{1}{2}}}{\frac{1}{2}(-2)}\right]_0^3 = -6\pi\left[0-9^{1/2}\right] = \underline{\underline{18\pi}}$$

terület e.

$$\mu(F_{gömb}) = 36\pi$$

$f(x,y,z)$ V tart.-on folyt.

osszuk fel $V_1, V_2, \dots, V_n$ résztartományokra hézagmentesen
minden pont csak egy tartományhoz tartozzon
legyen a résztartomány térfogata rendre.

$\mu(V_1), \mu(V_2), \dots, \mu(V_n)$

vegyünk fel minden résztartban egy $P_i(\xi_i, \eta_i, \zeta_i)$ pontot. $(i = 1, 2, \dots, n)$

majd képezzük.

$$t_n := \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i)\, \mu(V_i)$$

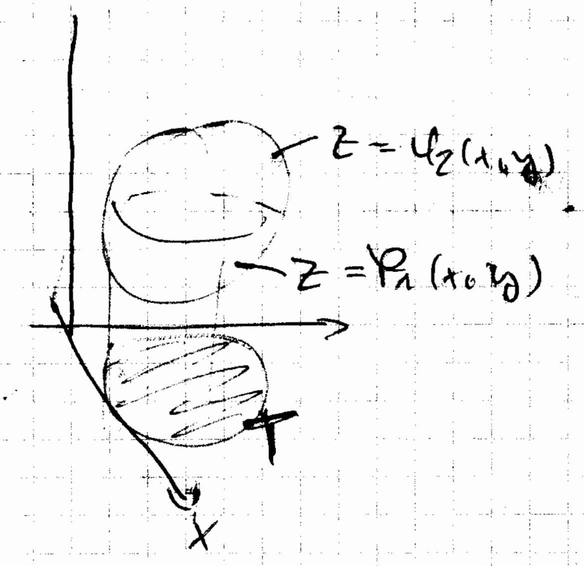
Def. a t_n int. köz. össz.

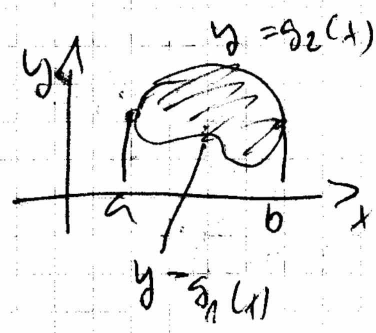
$$\iiint_V f(x,y,z)\, \overbrace{dx\, dy\, dz}^{dV\text{-elemi térfogat}}$$

Tul: hasonló mint a 2-ős integrálnál

Kiszámítás 3-szoros integrállal

$$= \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} \int_{z=\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\, dx\, dy\, dz =$$

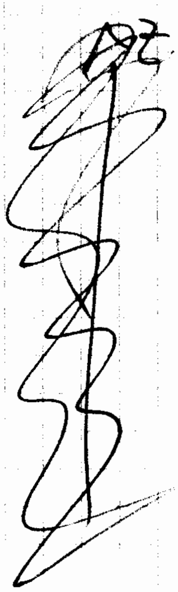




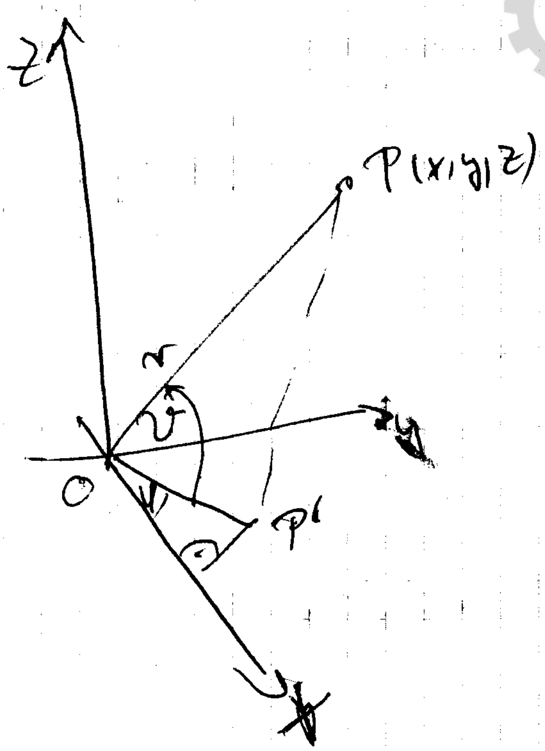
új változó bevezetése

hengerkoordinátarendszer

$$(x,y,z) \Rightarrow (r,\varphi,z)$$



Gömbi koordináta rendszer



$$(x,y,z) \to (r, \varphi, \vartheta)$$

$$\left.\begin{aligned} x &= r \cdot \cos\vartheta \cos\varphi \\ y &= r \cdot \cos\vartheta \cdot \sin\varphi \\ z &= r \cdot \sin\vartheta \end{aligned}\right\} \quad \left.\begin{aligned} -\infty < x < +\infty \\ -\infty < y < +\infty \\ -\infty < z < +\infty \end{aligned}\right\}$$

$$dx\,dy\,dz = \underbrace{r^2\cos\vartheta}_{|J|}\,dr\,d\varphi\,d\vartheta$$

$$\left.\begin{aligned} r &\geq 0 \\ 0 \leq \varphi &\leq 2\pi \\ -\frac{\pi}{2} \leq \vartheta &\leq \frac{\pi}{2} \end{aligned}\right\} \quad (-\pi \leq \varphi \leq \pi)$$

Alkalmazás:

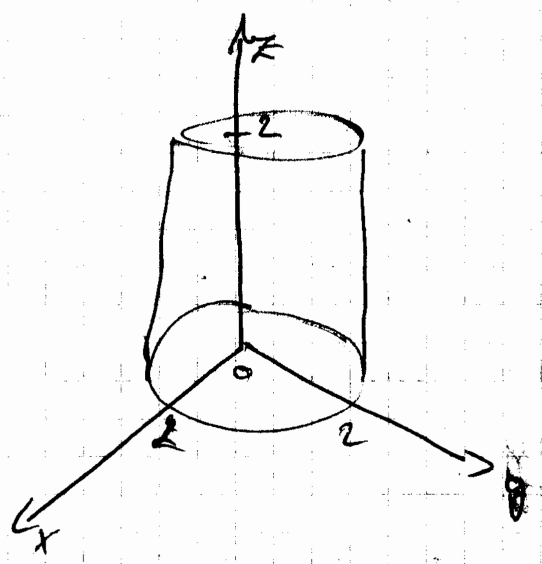
Térfogat:

$$\boxed{\mu(V) = \iiint_V dx\,dy\,dz}$$

$$\mu(V) \iiint_V dx\,dy\,dz = \int_{r=0}^{3}\int_{\varphi=0}^{2\pi}\int_{\vartheta=-\frac{\pi}{2}}^{\pi/2} r^2\cos\vartheta\,dr\,d\varphi\,d\vartheta =$$

$$= \int_{\varphi=0}^{2\pi}\int_{\vartheta=-\frac{\pi}{2}}^{\pi/2} \cos\vartheta \left[\frac{r^3}{3}\right]_{r=0}^{3} d\varphi\,d\vartheta = 9\int_{\varphi=0}^{2\pi}\left[\sin\vartheta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi =$$

$$= 9\int_{\varphi=0}^{2\pi}[1-(-1)]\,d\varphi = 18\,[\varphi]_0^{2\pi} = 36\pi$$



$$\iiint_V x\,dx\,dy\,dz = \left| \begin{array}{l} x = r\cos\varphi \\ y = r\cdot\sin\varphi \\ z = z \\ dx\,dy\,dz = r\,dr\,d\varphi\,dz \end{array} \right|$$

$x^2+y^2=4$

z tetsz

$$= \int_{\varphi=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{2} r\cdot\cos\varphi\; r\cdot dr\,d\varphi\,dz =$$

$$= \left[\frac{r^3}{3}\right]_0^2 \left[\sin\varphi\right]_0^{\pi} \left[z\right]_0^2 = \underline{\underline{0}}$$

Hf. a kidolgozott feladatok a jegyzetből