Feladatok I. éves levelező gépészmérnök hallgatóknak

1998/99. tanév II. félév

II. rész

Beadási határidő:

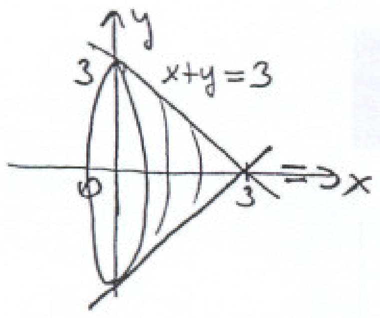
6. Számítsa ki az alábbi módon keletkező forgástestek térfogatát:
   - a) az $x+y=3$ egyenes első síknegyedbeli íve forog az x ill. y tengely körül;
   - b) az $xy=4$ görbe $1 \le x \le 4$ íve forog az x tengely;
   - c) az $y=\sin 2x$ görbe $0 \le x \le \frac{\pi}{6}$ íve forog az x tengely;
   - d) az $y^2=9-x$ görbe $0 \le x \le 9$ íve forog az x tengely;
   - e) az $2x^2+y^2=4$ ellipszis forog az x ill. y tengely körül;
   - f) az $y=x\left(x^2-1\right)$ görbe $-1 \le x \le 0$ íve forog az x tengely.
7. Számítsa ki a következő görbék ívhosszát:
   - a) $y=\ln(\sin x), \quad \frac{\pi}{6} \le x \le \frac{\pi}{3}$; b) $r=e^{3\varphi}, \quad 0 \le \varphi \le \pi$;
   - c) $x=e^t \cos t, \quad y=e^t \sin t, \quad 0 \le t \le \frac{\pi}{2}$.
8. Számítsa ki a következő görbék x tengely körüli forgásával keletkező forgásfelületek felszínét:
   - a) $y=\sqrt{x}, \quad 0 \le x \le 2$; b) $y=3x, \quad 0 \le x \le 3$.
9. Számítsa ki az alábbi kettős integrálokat:
   - a) $\displaystyle\int_{x=1}^{2}\int_{y=0}^{x}(x-2)y^2\,dxdy$; b) $\displaystyle\iint_{x^2+y^2\le 4} xy^2\,dxdy$.
10. Oldja meg az alábbi feladatokat henger- ill. gömbi koordináták bevezetésével és vázolja az integrációs tartományt:
    - a) $\displaystyle\int_{x=0}^{2}\int_{y=0}^{\sqrt{2x-x^2}}\int_{z=0}^{4} z\sqrt{x^2+y^2}\,dxdydz$; b) $\displaystyle\int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{z=0}^{\sqrt{9-x^2-y^2}}\left(x^2+y^2\right)dxdydz$;
11. Számítsa ki az alábbi felületek által határolt térrész térfogatát:
    - a) $z=4-x^2-y^2, \quad z=0$; b) $x^2+y^2+z^2=9, \quad 8z=x^2+y^2$;
    - c) $z=6-x^2-y^2, \quad z=\sqrt{x^2+y^2}$.
12. Számítsa ki integrálszámítás segítségével az $R=2$ sugarú gömb térfogatát és felszínét.
13. Számítsa ki a következő felületek felszínét:
    - a) $6x+3y+3z=24, \; x \ge 0, \; y \ge 0, \; z \ge 0$;
    - b) $x^2+y^2+z^2=9, \; x^2+y^2 \le 4, \; z \ge 0$.

1999/2000. tanév II. félév

Feladatok megoldása

II. rész

6. a.)

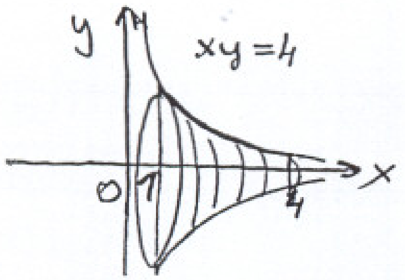


$$V_x = \pi\int_{x=0}^{3}(3-x)^2\,dx = \pi\left[\frac{(3-x)^3}{-3}\right]_0^3 =$$

$$= \frac{\pi}{-3}\left(0-3^3\right) = \underline{\underline{9\pi}}$$

$$V_y = \pi\int_{y=0}^{3}(3-y)^2\,dy = \pi\left[\frac{(3-y)^3}{-3}\right]_0^3 = \underline{\underline{9\pi}}$$

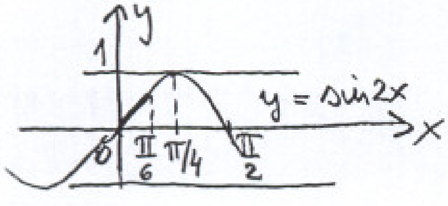
b.)



$$V_x = \pi\int_{x=1}^{4}\left(\frac{4}{x}\right)^2 dx = 16\pi\left[-\frac{1}{x}\right]_1^4 = 16\pi\left[-\frac{1}{4}+1\right] =$$

$$= 16\pi\cdot\frac{3}{4} = \underline{\underline{12\pi}}$$

c.)

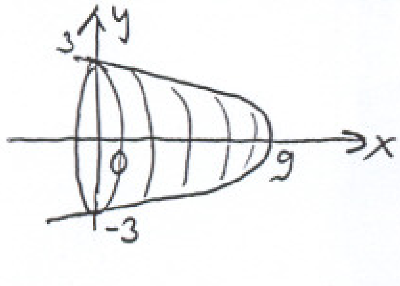


$$V_x = \pi\int_0^{\pi/6}\sin^2 2x\,dx = \pi\int_0^{\pi/6}\frac{1-\cos 4x}{2}\,dx =$$

$$= \frac{\pi}{2}\left[x-\frac{\sin 4x}{4}\right]_0^{\pi/6} = \frac{\pi}{2}\left[\frac{\pi}{6}-\frac{\sin\frac{2\pi}{3}}{4}\right] =$$

$$= \frac{\pi}{2}\left[\frac{\pi}{6}-\frac{\sqrt{3}}{8}\right] = \underline{\underline{\frac{\pi}{4}\left[\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right]}}$$

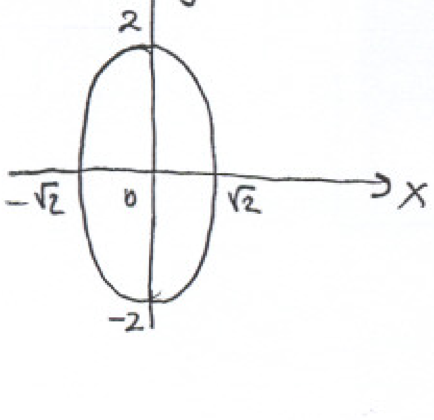
d.)



$$V_x = \pi\int_0^{9}(9-x)\,dx = \pi\left[9x-\frac{x^2}{2}\right]_0^9 =$$

$$= \pi\left[81-\frac{81}{2}\right] = \underline{\underline{\frac{81\pi}{2}}}$$

e.)

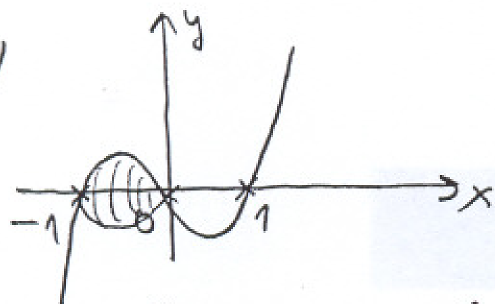


$$V_x = 2\pi\int_{x=0}^{\sqrt{2}}(4-2x^2)\,dx = 2\pi\left[4x-\frac{2x^3}{3}\right]_0^{\sqrt{2}} =$$

$$= 2\pi\left[4\sqrt{2}-\frac{2\cdot 2\sqrt{2}}{3}\right] = 2\pi\,\frac{2}{3}\,4\sqrt{2} = \underline{\underline{\frac{16\sqrt{2}\,\pi}{3}}}$$

⑥ e.) $$V_y = 2\pi \int_{y=0}^{2} \frac{1}{2}(4-y^2)\,dy = \pi\left[4y - \frac{y^3}{3}\right]_0^2 = \pi\left[8-\frac{8}{3}\right] = \underline{\underline{\frac{16\pi}{3}}}$$

⑥ f.)



$$y = x(x^2-1) = x(x-1)(x+1)$$

zh: $x=0$ (1)

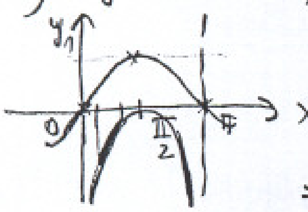
$x=\pm 1$ (1)

$$y_A = x^3$$

$$V_x = \pi \int_{-1}^{0} (x^3-x)^2\,dx = \pi \int_{-1}^{0} (x^6-2x^4+x^2)\,dx = \pi\left[\frac{x^7}{7} - \frac{2x^5}{5} + \frac{x^3}{3}\right]_{-1}^{0} =$$

$$= -\pi\left[-\frac{1}{7}+\frac{2}{5}-\frac{1}{3}\right] = -\pi\,\frac{-15+42-35}{105} = \underline{\underline{\frac{8\pi}{105}}}$$

⑦ a.) $y=\ln(\sin x)$; $\frac{\pi}{6} \le x \le \frac{\pi}{3}$; $$s = \int_{\pi/6}^{\pi/3} \frac{1}{\sin x}\,dx = \int_{\pi/6}^{\pi/3} \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx =$$



$$1+(y')^2 = 1+\left(\frac{\cos x}{\sin x}\right)^2 = \frac{1}{\sin^2 x}$$

$$= \int_{\pi/6}^{\pi/3} \left(\frac{\sin\frac{x}{2}}{2\cos\frac{x}{2}} + \frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)dx = \left[-\ln\left(\cos\frac{x}{2}\right) + \ln\left(\sin\frac{x}{2}\right)\right]_{\pi/6}^{\pi/3} =$$

$$= \left[\ln\left(\operatorname{tg}\frac{x}{2}\right)\right]_{\pi/6}^{\pi/3} = \ln\left(\operatorname{tg}\frac{\pi}{6}\right) - \ln\left(\operatorname{tg}\frac{\pi}{12}\right)$$

b.) $r = e^{3\varphi}$; $0 \le \varphi \le \pi$; $$r^2 + (r')^2 = e^{6\varphi} + (3e^{3\varphi})^2 = 10e^{6\varphi}$$

$$s = \int_0^{\pi} \sqrt{10e^{6\varphi}}\,d\varphi = \int_0^{\pi} \sqrt{10}\,e^{3\varphi}\,d\varphi = \left[\frac{\sqrt{10}}{3}e^{3\varphi}\right]_0^{\pi} =$$

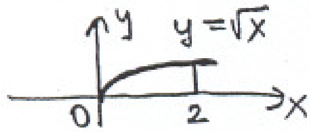
$$= \underline{\underline{\frac{\sqrt{10}}{3}\left[e^{3\pi}-1\right]}}$$

c.) $\left.\begin{array}{l} x = e^t\cos t \\ y = e^t \sin t \end{array}\right\}$ $0 \le t \le \frac{\pi}{2}$ $$\dot{x}^2 + \dot{y}^2 = e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 =$$

$$= e^{2t}\left[\cos^2 t + \sin^2 t - \cancel{2\cos t \sin t} + \sin^2 t + \cos^2 t + \cancel{2\sin t\cos t}\right] = 2e^{2t};$$

$$s = \int_0^{\pi/2} \sqrt{2}e^t\,dt = \sqrt{2}\left[e^t\right]_0^{\pi/2} = \underline{\underline{\sqrt{2}\left[e^{\pi/2}-1\right]}}$$

⑧ a.)



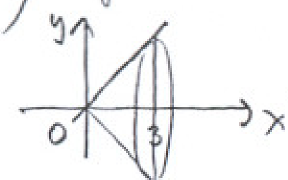
$0 \le x \le 2$; $$1+[y'(x)]^2 = 1+\left[\frac{1}{2\sqrt{x}}\right]^2 = 1+\frac{1}{4x}$$

$$F_x = 2\pi \int_0^2 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx = 2\pi\int_0^2 \sqrt{x+\frac{1}{4}}\,dx =$$

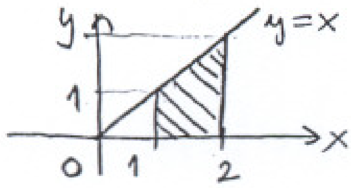
$$= 2\pi\left[\frac{2\left(x+\frac{1}{4}\right)^{3/2}}{3}\right]_0^2 = \frac{4\pi}{3}\left[\left(2+\frac{1}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right] = \frac{4\pi}{3}\left[\left(\frac{9}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right] =$$

$$= \frac{4\pi}{3}\left[\frac{27}{8} - \frac{1}{8}\right] = \frac{4\pi \cdot 26}{3 \cdot 8} = \underline{\underline{\frac{13\pi}{3}}}$$

8.) b.) $y = 3x$ $\quad F_x = 2\pi\int_{x=0}^{3} 3x\sqrt{1+3^2}\,dx = 2\sqrt{10}\,\pi\left[\frac{3x^2}{2}\right]_0^3 = \underline{\underline{27\sqrt{10}\,\pi}}$



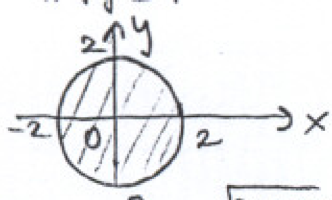
9.) a.) $\int_{x=1}^{2}\int_{y=0}^{x}(x-2)y^2\,dx\,dy = \int_{x=1}^{2}(x-2)\left[\frac{y^3}{3}\right]_0^x dx = \frac{1}{3}\int_{x=1}^{2}x^3(x-2)\,dx =$



$= \frac{1}{3}\left[\frac{x^5}{5}-\frac{x^4}{2}\right]_1^2 = \frac{1}{3}\left[\frac{32}{5}-8-\frac{1}{5}+\frac{1}{2}\right] =$

$= \frac{1}{3}\left(\frac{31}{5}-\frac{15}{2}\right) = \frac{1}{3}\,\frac{62-75}{10} = \underline{\underline{-\frac{13}{30}}}$

b.) $\iint\limits_{x^2+y^2\le 4} xy^2\,dx\,dy = \int_{r=0}^{2}\int_{\varphi=0}^{2\pi} r\cos\varphi\, r^2\sin^2\varphi\, r\,dr\,d\varphi = \left[\frac{r^5}{5}\right]_0^2\left[\frac{\sin^3\varphi}{3}\right]_0^{2\pi} = \underline{\underline{0}}$



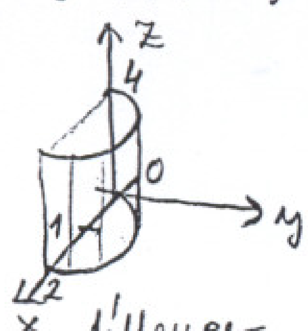
10.) a.) $\int_{x=0}^{2}\int_{y=0}^{\sqrt{2x-x^2}}\int_{z=0}^{4} z\sqrt{x^2+y^2}\,dx\,dy\,dz = \int_{r=0}^{2\cos\varphi}\int_{\varphi=0}^{\pi/2}\int_{z=0}^{4} z\,r^2\,dr\,d\varphi\,dz$

$y=\sqrt{2x-x^2} \Rightarrow \underbrace{y^2+x^2}_{r^2}-2x=0 \Rightarrow (x-1)^2+y^2=1$

$r^2-2r\cos\varphi=0$

$\Rightarrow \boxed{r=2\cos\varphi}$

$= \left|\begin{array}{l} x=r\cos\varphi \\ y=r\sin\varphi \\ z=z \\ dx\,dy\,dz=r\,dr\,d\varphi\,dz \end{array}\right| =$



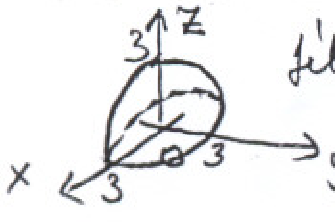
félhenger

$= \int_{\varphi=0}^{\pi/2}\int_{r=0}^{2\cos\varphi}\left[\frac{z^2}{2}\right]_0^4 r^2\,dr\,d\varphi = 8\int_{\varphi=0}\left[\frac{r^3}{3}\right]_0^{2\cos\varphi} d\varphi =$

$= \frac{8}{3}\int_{\varphi=0}^{\pi/2} 8\underbrace{\cos^3\varphi}_{\cos\varphi(1-\sin^2\varphi)}\,d\varphi = \frac{64}{3}\left[\sin\varphi-\frac{\sin^3\varphi}{3}\right]_0^{\pi/2} =$

$= \frac{64}{3}\left[1-\frac{1}{3}\right] = \underline{\underline{\frac{128}{9}}}$

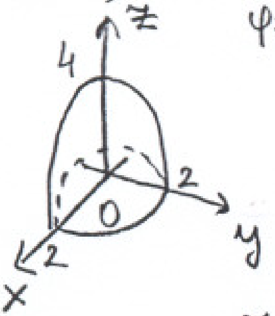
b.) $\int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{z=0}^{\sqrt{9-x^2-y^2}}(x^2+y^2)\,dx\,dy\,dz = \left|\begin{array}{l} x=r\cos\vartheta\cos\varphi \\ y=r\cos\vartheta\sin\varphi \\ z=r\sin\vartheta \\ dx\,dy\,dz=r^2\cos\vartheta\,dr\,d\varphi\,d\vartheta \end{array}\right|$



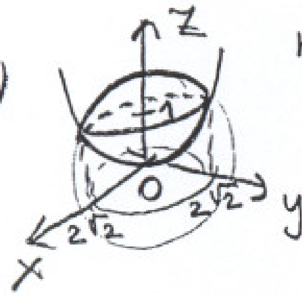
félgömb

$= \int_{r=0}^{3}\int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi/2} r^2(\cos^2\vartheta\cos^2\varphi+\cos^2\vartheta\sin^2\varphi)\,r^2\cos\vartheta\cdot dr\,d\varphi\,d\vartheta =$

11.) a.) $V=\int\limits_{\varphi=0}^{2\pi}\int\limits_{r=0}^{2}(4-r^2)\,r\,dr\,d\varphi = 2\pi\left[2r^2-\frac{r^4}{4}\right]_0^2 = 2\pi[8-4]=\underline{\underline{8\pi}}$



b.)



metszésvonal:

$$\left.\begin{aligned} x^2+y^2+z^2&=9\\ 8z&=x^2+y^2 \end{aligned}\right\}$$

$$z^2+8z-9=0$$

$$z_{12}=\frac{-8\pm\sqrt{64+36}}{2}=\frac{-8\pm10}{2}=\begin{cases}1\\ \cancel{-9}\end{cases}$$

Tehát a $z=1$ síkban az $x^2+y^2=8$ kör.

$\Downarrow$

$$V=\int\limits_{\varphi=0}^{2\pi}\int\limits_{r=0}^{2\sqrt{2}}\int\limits_{z=\frac{r^2}{8}}^{\sqrt{9-r^2}} r\,dr\,d\varphi\,dz = 2\pi\int\limits_{r=0}^{2\sqrt{2}}\left(\sqrt{9-r^2}-\frac{r^2}{8}\right)r\,dr =$$

$$=2\pi\left[\frac{(9-r^2)^{3/2}}{\frac{3}{2}(-2)}-\frac{r^4}{32}\right]_0^{2\sqrt{2}} = 2\pi\left[\frac{1}{-3}-\frac{16\cdot4}{32}-\frac{9\cdot3}{-3}\right]=$$

$$=2\pi\left[-\frac{1}{3}-2+9\right]=2\pi\left[7-\frac{1}{3}\right]=\underline{\underline{\frac{40\pi}{3}}}$$

c.)

$$\left.\begin{aligned} z&=6-x^2-y^2\\ z&=\sqrt{x^2+y^2} \end{aligned}\right\} \Rightarrow z=6-z^2$$

$$z^2+z-6=0$$

$$z_{12}=\frac{-1\pm\sqrt{1+24}}{2}=\frac{-1\pm5}{2}=\begin{cases}2\\ -3\end{cases}$$

A metszésvonal a $z=2$ síkban az $x^2+y^2=4$ kör

$$V=\int\limits_{r=0}^{2}\int\limits_{\varphi=0}^{2\pi}\left[6-r^2-r\right]r\,dr\,d\varphi = 2\pi\left[3r^2-\frac{r^4}{4}-\frac{r^3}{3}\right]_0^2=$$

$$=2\pi\left[12-4-\frac{8}{3}\right]=2\pi\left[8-\frac{8}{3}\right]=\underline{\underline{\frac{32\pi}{3}}}$$

(12) $V = 2\int\limits_{r=0}^{2}\int\limits_{\varphi=0}^{2\pi}\sqrt{4-r^2}\,r\,dr\,d\varphi = 2\cdot 2\pi\left[\frac{(4-r^2)^{3/2}}{\frac{3}{2}(-2)}\right]_0^2 =$

a gömb egyenlete: $x^2+y^2+z^2=4 \Rightarrow z^2=4-x^2-y^2$

$= 4\pi\left[0-\frac{4\cdot 2}{-3}\right] = \underline{\underline{\frac{32\pi}{3}}}$

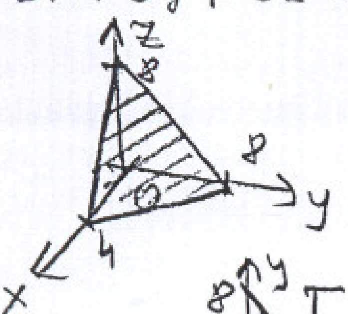
$F = 2\int\limits_{r=0}^{2}\int\limits_{\varphi=0}^{2\pi}\frac{2}{\sqrt{4-r^2}}\,r\,dr\,d\varphi = 8\pi\left[\frac{(4-r^2)^{\frac{1}{2}}}{\frac{1}{2}(-2)}\right]_0^2 =$

$dF = \sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy = \sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}} = \frac{2}{\sqrt{4-x^2-y^2}}$

$z = \sqrt{4-x^2-y^2}$

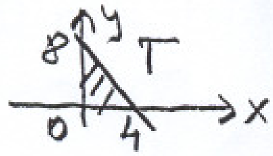
$= 8\pi\left[0-\frac{2}{-1}\right] = \underline{\underline{16\pi}}$

(13.) a) $6x+3y+3z=24;\quad x\geq 0;\ y\geq 0;\ z\geq 0$



$F = \iint\limits_T dF = \int\limits_{x=0}^{4}\int\limits_{y=0}^{8-2x}\sqrt{1+4+1}\,dx\,dy =$

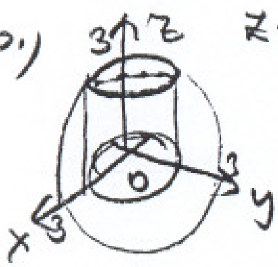
$z = 8-2x-y$



$= \sqrt{6}\int\limits_{x=0}^{4}(8-2x)\,dx = \sqrt{6}\left[8x-x^2\right]_0^4 =$

$= \sqrt{6}\left[32-16\right] = \underline{\underline{16\sqrt{6}}}$

b)



$z = \sqrt{9-x^2-y^2}$ $\quad F = \iint\limits_T \sqrt{1+\frac{x^2+y^2}{9-x^2-y^2}}\,dx\,dy =$

$= \int\limits_{r=0}^{2}\int\limits_{\varphi=0}^{2\pi}\frac{3}{\sqrt{9-r^2}}\,r\,dr\,d\varphi = 3\cdot 2\pi\left[\frac{(9-r^2)^{\frac{1}{2}}}{\frac{1}{2}(-2)}\right]_0^2 =$

$= -6\pi\left[\sqrt{5}-3\right] = \underline{\underline{6\pi\left[3-\sqrt{5}\right]}}$