Feladatok I. éves levelező gépészmérnök hallgatóknak

1999/2000. tanév II. félév

I. rész

Beadási határidő:

1. Számítsa ki az alábbi függvények deriváltját:

a) $f(x,y) = x\sin\left(y^2 + e^{2x}\right) + y\ln(2x - y);$ $\quad f'_x = ?;\quad f'_y = ?;$

b) $f(x,y) = arctg(x+y) + xy^2 + y3^{2x};$ $\quad f''_{xx} = ?;$

c) $f(x,y) = \arcsin\sqrt{2x^2 - y} + \ln xy - 1;$ $\quad f'_x = ?,\quad f'_y = ?;$

d) $f(x,y) = 4y^3x^2\cos x + 7x\sin 2y;$ $\quad f''_{xy} = ?;\quad f''_{yy} = ?.$

2. Vizsgálja meg, hogy az alábbi függvényeknek hol lehet szélsőértéke, és ha van, milyen ez a szélsőérték?

a) $z = x^2 - xy + y^2 - 1;$ b) $z = x^2 + xy + y^2 + y + \frac{1}{3};$ –

c) $z = \frac{1}{x} + \frac{1}{y} + \frac{xy}{27};$ d) $z = x^3 - 3xy + y^3.$

3. Számítsa ki az alábbi integrálokat:

a) $\int \frac{xdx}{x^2 - 5x + 6};$ b) $\int x\cos 2xdx;$ c) $\int x^2e^{4x}dx;$

d) $\int \frac{1}{x^2 + 16x + 60}dx;$ e) $\int \cos^2 4xdx;$ f) $\int \ln xdx;$

g) $\int \frac{xdx}{x^2 + x - 6};$ h) $\int_1^2 x\ln xdx;$ i) $\int_0^{\frac{\pi}{2}} \sin^2 2xdx;$

j) $\int \frac{x-3}{x^3 - 4x^2 + 4x}dx;$ k) $\int x^2e^{-x}dx;$ l) $\int_3^{\infty} \frac{1}{x-2}dx;$

m) $\int_0^2 \frac{dx}{x^2 - 4x + 3};$ n) $\int_1^2 \frac{dx}{x\ln x};$ o) $\int_1^{\infty} \frac{1}{x^2}dx.$

4. Számítsa ki az

a) $y = \sin x$ görbe $0 \le x \le \pi$ íve és az x tengely által

b) $y = 3\cos x$ görbe $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ íve és az x tengely által

c) $y = \frac{x}{x^2+1}$ görbe és az $y = \frac{x}{5}$ egyenes által határolt első síknegyedben

d) $y = x^2$ és az $x + y = 2$ görbék által

közrezárt síkrész területét!

5. Számítsa ki az alábbi görbék által határolt területet:

a) $r = 3\cos\varphi;$ b) $r = 1 - \cos\varphi;$

c) $x = 2\cos t,\quad y = 2\sin t;$ d) $x = 9\cos t,\quad y = 4\sin t.$

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Feladatok megoldása

I. rész

1. a.) $f'_x = \sin(y^2+e^{2x}) + x\cos(y^2+e^{2x})\cdot 2e^{2x} + \frac{2y}{2x-y}$;

$f'_y = x\cos(y^2+e^{2x})\cdot 2y + \ln(2x-y) - \frac{y}{2x-y}$

b.) $f'_x = \frac{1}{1+(x+y)^2} + y^2 + 2y\cdot 3^{2x}\ln 3$;

$f''_{xx} = \frac{-2(x+y)}{[1+(x+y)^2]^2} + 4y\cdot 3^{2x}(\ln 3)^2$;

c.) $f'_x = \frac{1}{\sqrt{1-(2x^2-y)}}\cdot\frac{4x}{2\sqrt{2x^2-y}} + \frac{1}{x}$;

$f'_y = \frac{1}{\sqrt{1-(2x^2-y)}}\cdot\frac{-1}{2\sqrt{2x^2-y}} + \frac{1}{y}$;

d.) $f'_x = 8xy^3\cos x - 4y^3x^2\sin x + 7\sin 2y$;

$f''_{xy} = 24xy^2\cos x - 12y^2x^2\sin x + 14\cos 2y$;

$f'_y = 12y^2x^2\cos x + 14x\cos 2y$;

$f''_{yy} = 24yx^2\cos x - 28x\sin 2y$;

2. a.) $z = x^2 - xy + y^2 - 1$

$$\left.\begin{array}{l} z'_x = 2x - y = 0 \\ z'_y = -x + 2y = 0 \end{array}\right\} \begin{array}{l} y = 2x \\ \Rightarrow 3x = 0 \\ \boxed{x = 0 = y} \end{array}$$

$P(0,0)$

$z''_{xx} = 2$ $\quad z''_{yy} = 2$

$z''_{xy} = -1$

$D(P) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$

van szé.

$z''_{xx}(P) > 0 \Rightarrow$ min.

$\boxed{P_{min.}(0;0;-1)}$

2.) b.) $z = x^2 + xy + y^2 + y + \frac{1}{3}$

$$\left.\begin{aligned} z'_x &= 2x + y = 0 \\ z'_y &= x + 2y + 1 = 0 \end{aligned}\right\} \Rightarrow y = -2x$$

$x - 4x + 1 = 0$

$-3x = -1$

$\underline{\underline{x = \frac{1}{3}}}; \quad \underline{\underline{y = -\frac{2}{3}}}$ $\quad P\left(\frac{1}{3}; -\frac{2}{3}\right)$ - ban lehet szé.

$z''_{xx} = 2$

$z''_{xy} = 1$

$z''_{yy} = 2$

$D(P) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$ van szé. $\quad z''_{xx} = 2 > 0 \Rightarrow$ min.

$$\boxed{P_{min.}\left(\frac{1}{3}; -\frac{2}{3}; 0\right)}$$

c.) $z = \frac{1}{x} + \frac{1}{y} + \frac{xy}{27}$ $\quad (x \neq 0;\ y \neq 0)$

$$\left.\begin{aligned} z'_x &= -\frac{1}{x^2} + \frac{y}{27} = 0 \\ z'_y &= -\frac{1}{y^2} + \frac{x}{27} = 0 \end{aligned}\right\} \Rightarrow y = \frac{27}{x^2}$$

$\Rightarrow -\frac{x^4}{27^2} + \frac{x}{27} = 0 \Rightarrow \frac{x}{27}\left(1 - \frac{x^3}{27}\right) = 0$

$x_1 = 3 \quad y_1 = 3 \quad \underline{P_1(3; 3)}$

$\cancel{x_2 = 0}$

$z''_{xx} = \frac{2}{x^3}$

$z''_{xy} = \frac{1}{27}$

$z''_{yy} = \frac{2}{y^3}$

$D = \begin{vmatrix} \frac{2}{x^3} & \frac{1}{27} \\ \frac{1}{27} & \frac{2}{y^3} \end{vmatrix} = \frac{4}{x^3 y^3} - \frac{1}{27^2}$

$D(P_1) = \frac{4}{27^2} - \frac{1}{27^2} = \frac{3}{27^2} > 0$ van szé.

$z''_{xx} = \frac{2}{27} > 0$ min. $\quad z_{min} = 1$

$$\boxed{P_{min}(3; 3; 1)}$$

d.) $z = x^3 - 3xy + y^3$

$$\left.\begin{aligned} z'_x &= 3x^2 - 3y = 0 \\ z'_y &= -3x + 3y^2 = 0 \end{aligned}\right\} \begin{aligned} &\Rightarrow y = x^2 \\ &\Rightarrow x^4 - x = 0 \end{aligned}$$

$x(x-1)(x^2+x+1) = 0$

$x_1 = 0; \quad y_1 = 0; \quad P_1(0; 0)$

$x_2 = 1; \quad y_2 = 1; \quad P_2(1; 1)$

$$D = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 36xy - 9; \qquad D(P_1) = -9 < 0 \quad \text{nincs szé}$$

$$D(P_2) = 27 > 0 \quad \text{van szé}$$

$$z''_{xx}(P_2) = 6 > 0 \quad \text{min.} \qquad \underline{z_{min} = -1}$$

$$\boxed{P_{min}(1; 1; -1)}$$

3. a.) $\int \frac{x\,dx}{x^2-5x+6} = \int \left(\frac{3}{x-3} - \frac{2}{x-2}\right) dx = 3\ln|x-3| - 2\ln|x-2| + C$

$$\frac{x}{x^2-5x+6} = \frac{x}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} = \frac{A(x-2)+B(x-3)}{(\;)(\;)}$$

$$x = A(x-2) + B(x-3) \qquad \begin{matrix} x=2 \\ x=3 \end{matrix} \quad \boxed{\begin{matrix} B=-2 \\ A=3 \end{matrix}}$$

b.) $\int x\cos 2x\,dx = \begin{vmatrix} u=x & v'=\cos 2x \\ u'=1 & v=\frac{\sin 2x}{2} \end{vmatrix} = \frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}\,dx =$

$$= \frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + C$$

c.) $\int x^2 e^{4x}\,dx = \begin{vmatrix} u=x^2 & v'=e^{4x} \\ u'=2x & v=\frac{e^{4x}}{4} \end{vmatrix} = \frac{x^2e^{4x}}{4} - \frac{1}{2}\int xe^{4x}\,dx =$

$$= \begin{vmatrix} u=x & v'=e^{4x} \\ u'=1 & v=\frac{e^{4x}}{4} \end{vmatrix} = \frac{x^2e^{4x}}{4} - \frac{1}{2}\left[\frac{xe^{4x}}{4} - \int \frac{e^{4x}}{4}dx\right] = \frac{x^2e^{4x}}{4} - \frac{xe^{4x}}{8} + \frac{e^{4x}}{32} + C$$

d.) $\int \frac{1}{x^2+16x+60}\,dx = \int \left(\frac{\frac{1}{4}}{x+6} - \frac{\frac{1}{4}}{x+10}\right) dx = \frac{1}{4}\ln|x+6| - \frac{1}{4}\ln|x+10| + C$

$$= \frac{1}{4}\ln\left|\frac{x+6}{x+10}\right| + C$$

e.) $\int \cos^2 4x\,dx = \int \frac{1+\cos 8x}{2}\,dx = \frac{1}{2}\left[x + \frac{\sin 8x}{8}\right] + C$

f.) $\int \ln x\,dx = \begin{vmatrix} u=\ln x & v'=1 \\ u'=\frac{1}{x} & v=x \end{vmatrix} = x\ln x - \int dx = x\ln x - x + C$

g.) $\int \frac{x\,dx}{x^2+x-6} = \int \left(\frac{\frac{3}{5}}{x+3} + \frac{\frac{2}{5}}{x-2}\right) dx = \frac{3}{5}\ln|x+3| + \frac{2}{5}\ln|x-2| + C$

$$\frac{x}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} \Rightarrow x = A(x-2) + B(x+3)$$

$$\begin{matrix} x=2 & 2=5B \\ x=-3 & -3=-5A \end{matrix}$$

③ h.) $\int_1^2 x\ln x\,dx = \left|\begin{matrix} u=\ln x & v'=x \\ u'=\frac{1}{x} & v=\frac{x^2}{2}\end{matrix}\right| = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x}{2}\,dx =$

$= \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_1^2 = 2\ln 2 - 1 + \frac{1}{4} = \underline{\ln 4 - \frac{3}{4}}$

i.) $\int_0^{\pi/2} \sin^2 2x\,dx = \int_0^{\pi/2} \frac{1-\cos 4x}{2}\,dx = \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}$

j.) $\int \frac{x-3}{x^3-4x^2+4x}\,dx = \int\left(\frac{-\frac{3}{4}}{x} + \frac{\frac{3}{4}}{x-2} + \frac{-\frac{1}{2}}{(x-2)^2}\right)dx = -\frac{3}{4}\ln|x| + \frac{3}{4}\ln|x-2| +$

$\frac{x-3}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \Rightarrow x-3 = A(x-2)^2 + Bx(x-2) + Cx$

$x=2 \quad -1 = 2C \Rightarrow \boxed{C=-\frac{1}{2}} \qquad x^2:\ 0=A+B$

$x=0 \quad -3 = 4A \Rightarrow \boxed{A=-\frac{3}{4}} \qquad \boxed{B=\frac{3}{4}}$

$+ \frac{1}{2}\frac{1}{x-2} + C = \ln\sqrt[4]{\left|\frac{x-2}{x}\right|^3} + \frac{1}{2(x-2)} + C$

k.) $\int x^2 e^{-x}\,dx = \left|\begin{matrix} u=x^2 & v'=e^{-x} \\ u'=2x & v=-e^{-x}\end{matrix}\right| = -x^2e^{-x} + \int 2xe^{-x}\,dx =$

$= \left|\begin{matrix} u=2x & v'=e^{-x} \\ u'=2 & v=-\dots\end{matrix}\right| = -x^2e^{-x} - 2xe^{-x} + \int 2e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

l.) $\int_3^{+\infty} \frac{1}{x-2}\,dx = \lim_{b\to+\infty} \int_3^b \frac{1}{x-2}\,dx = \lim_{b\to+\infty} \left[\ln|x-2|\right]_3^b =$

$= \lim_{b\to+\infty} [\ln|b-2| - \underbrace{\ln 1}_{=0}] = \infty$ divergens.

m.) $\int_0^2 \frac{dx}{x^2-4x+3} = \int_0^2 \left(\frac{\frac{1}{2}}{x-3} - \frac{\frac{1}{2}}{x-1}\right)dx = \int_0^2 \frac{\frac{1}{2}}{x-3}\,dx + \int_0^2 \frac{-\frac{1}{2}}{x-1}\,dx =$ ⊛

$\frac{1}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x-3)$

$x=1 \quad B=-\frac{1}{2}$

$x=3 \quad A=\frac{1}{2}$

$\int_0^2 \frac{-\frac{1}{2}}{x-1}\,dx = \lim_{\varepsilon_1\to 0} \int_0^{1-\varepsilon_1} \frac{-\frac{1}{2}}{x-1}\,dx + \lim_{\varepsilon_2\to 0} \int_{1+\varepsilon_2}^2 \frac{-\frac{1}{2}}{x-1}\,dx =$

$= \lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}} \left[-\frac{1}{2}\left[\ln|x-1|\right]_0^{1-\varepsilon_1} - \frac{1}{2}\left[\ln|x-1|\right]_{1+\varepsilon_2}^2\right] = \underbrace{\lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}}\left[-\frac{1}{2}\underbrace{\ln\varepsilon_1}_{\downarrow\,-\infty} + \frac{1}{2}\underbrace{\ln\varepsilon_2}_{\downarrow\,-\infty}\right] \Rightarrow$

nem létezik a határérték

⊛ az impr. int.-nak nincs határértéke.

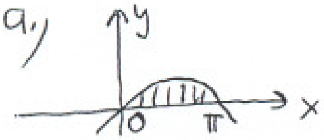
3.) a.) $\int_1^2 \frac{dx}{x\ln x} = \lim_{\varepsilon\to 0}\int_{1+\varepsilon}^{2}\frac{dx}{x\ln x} = \lim_{\varepsilon\to 0}\left[\ln|\ln x|\right]_{1+\varepsilon}^{2} = \lim_{\varepsilon\to 0}\left[\ln|\ln 2| - \underbrace{\ln(\ln(1+\varepsilon))}_{\to -\infty}\right]$

$x>0,\ \ln x \neq 0 \Rightarrow x \neq 1$

$= \infty$ divergens

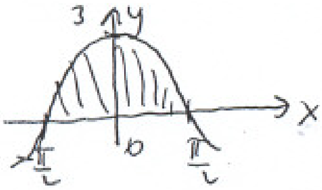
b.) $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left[-\underbrace{\frac{1}{b}}_{\to 0}+1\right] = 1$

4.) a.)



$T=\int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi = -\cos\pi + \cos 0 = 2$

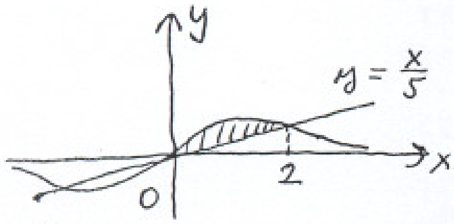
b.)



$T = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\cos x\,dx = 3\left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 3\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right] =$

$= 3(1-(-1)) = 6$

c.)



$\left.\begin{array}{l} y = \frac{x}{x^2+1} \\ y = \frac{x}{5} \end{array}\right\} \Rightarrow \frac{x}{x^2+1} = \frac{x}{5}$

$x\left(\frac{1}{x^2+1} - \frac{1}{5}\right) = 0$

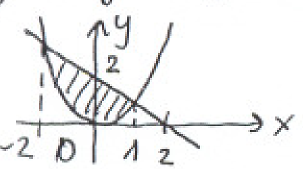
$x_1 = 0 \qquad 5 - x^2 - 1 = 0$

$x^2 = 4 \Rightarrow x_{2,3} = \pm 2$

$T = \int_{x=0}^{2}\left(\frac{x}{x^2+1} - \frac{x}{5}\right)dx = \left[\frac{1}{2}\ln(x^2+1) - \frac{x^2}{10}\right]_0^2 =$

$= \frac{1}{2}\ln 5 - \frac{4}{10} = \ln\sqrt{5} - \frac{2}{5}$

d.) $y = x^2;\quad x+y=2$



$\left.\begin{array}{l} y = x^2 \\ x+y=2 \end{array}\right\} -$

$-x = x^2 - 2$

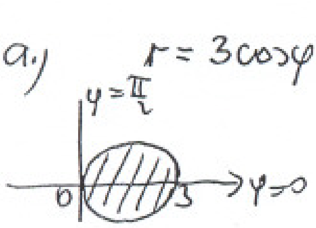
$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$T = \int_{x=-2}^{1}(2-x-x^2)\,dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} =$

$= 2 - \frac{1}{2} - \frac{1}{3} - \left(-4 - 2 + \frac{8}{3}\right) = \frac{27}{6} = 4{,}5$

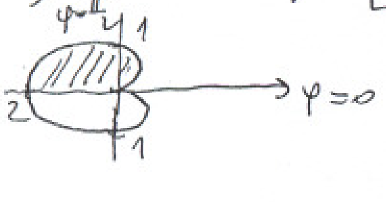
5.) a.) $r = 3\cos\varphi$ $\boxed{\text{kör}}$



$S = \frac{1}{2}\cdot 2\int_{\varphi=0}^{\pi/2} 9\cos^2\varphi\,d\varphi = 9\int_{\varphi=0}^{\pi/2}\frac{1+\cos 2\varphi}{2}\,d\varphi =$

$= \frac{9}{2}\left[\varphi + \frac{\sin 2\varphi}{2}\right]_0^{\pi/2} = \frac{9}{2}\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - 0\right] = \frac{9\pi}{4}$

b.) $r = 1-\cos\varphi$ $\boxed{\text{kardioid}}$



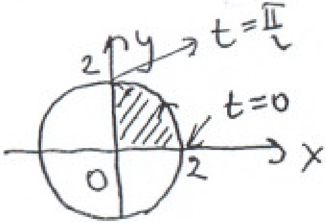
$S = \frac{1}{2}\cdot 2\int_{\varphi=0}^{\pi}(1-\cos\varphi)^2\,d\varphi =$

$= \int_{\varphi=0}^{\pi}\left(1 - 2\cos\varphi + \cos^2\varphi\right)d\varphi = \int_{\varphi=0}^{\pi}\left(1 - 2\cos\varphi + \frac{1+\cos 2\varphi}{2}\right)d\varphi$

5.) b.) $= \left[\varphi - 2\sin\varphi + \frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right]_0^{\pi} = \pi - 2\sin\pi + \frac{\pi}{2} + \frac{\sin 2\pi}{4} = \underline{\underline{\frac{3\pi}{2}}}$

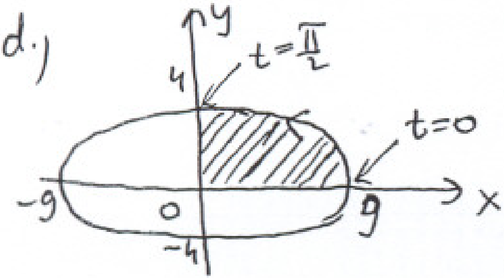
5.) c.) $\left.\begin{array}{l} x = 2\cos t, \\ y = 2\sin t \end{array}\right\}$

$$S = \frac{1}{2} \cdot 4 \int_{t=0}^{\pi/2} \left(2\cos t \cdot 2\cos t - 2\sin t \cdot (-2\sin t)\right) dt =$$



$$= 2 \int_{t=0}^{\pi/2} 4\,dt = 8 \cdot [t]_0^{\pi/2} = \underline{\underline{4\pi}}$$

5.) d.)



$$S = \frac{1}{2} \cdot 4 \int_{t=0}^{\pi/2} \left(9\cos t \cdot 4\cos t - 4\sin t(-9\sin t)\right) dt =$$

$$= 2 \int_{t=0}^{\pi/2} 36\,dt = 72[t]_0^{\pi/2} = \underline{\underline{36\pi}}$$

$\left.\begin{array}{l} x = 9\cos t \\ y = 4\sin t \end{array}\right\}$