Inegalităţi remarcabile

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$$a+\frac{1}{a}\geq 2, \forall a\in R_+^*$$

$$a^2+b^2+c^2\geq ab+bc+ca, \forall a,b,c\in R$$

Inegalitatea mediilor

$$m_h\leq m_g\leq m_a$$

$$\frac{2}{\frac{1}{a}+\frac{1}{b}}\leq\sqrt{ab}\leq\frac{a+b}{2}, \forall a,b>0$$

Egalitatea are loc dacă şi numai dacă a=b

Generalizare

$$\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}\leq\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}\leq\frac{a_1+a_2+\ldots+a_n}{2}, \forall a_1,a_2,\ldots,a_n>0,$$

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Cauchy – Buniakovski - Schwarz

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_1^2 + ... + b_n^2) \geq (a_1 b_1 + a_2 b_2 + ... + a_n b_n)^2$$

$$\forall n \in N^*, a_1, b_1, a_2, b_2, ..., a_n, b_n \in R$$

Minkowski

$$\sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2} \geq \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2}$$

$$\forall a_1, b_1, a_2, b_2 \in R$$

Bernoulli

$$(1 + x)^n \geq 1 + nx, \forall x \geq -1, n \in N^*$$

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Ex1: Dacă x,y,z sunt numere reale pozitive, să se arate că:

$$(x+y)(y+z)(z+x) \geq 8xyz$$

Dem.

$$x+y \geq 2\sqrt{xy}$$

$$y+z \geq 2\sqrt{yz}$$

$$z+x \geq 2\sqrt{zx}$$

Inegalitatea mediilor

$$(x+y)(y+z)(z+x) \geq 2\sqrt{xy} \cdot 2\sqrt{yz} \cdot 2\sqrt{zx}$$

$$(x+y)(y+z)(z+x) \geq 8\sqrt{x^2y^2z^2}$$

$$(x+y)(y+z)(z+x) \geq 8xyz$$

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Ex2: Dacă x,y,z sunt numere reale pozitive, să se arate că:

$$x+y+z \geq \sqrt{xy}+\sqrt{yz}+\sqrt{zx}$$

Dem.

$$x+y \geq 2\sqrt{xy}$$

$$y+z \geq 2\sqrt{yz}$$

$$z+x \geq 2\sqrt{zx}$$

Inegalitatea mediilor

$$-----------------------$$

$$2x+2y+2z \geq 2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}$$

$$x+y+z \geq \sqrt{xy}+\sqrt{yz}+\sqrt{zx}$$

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Ex3: Dacă x,y sunt numere reale pozitive şi xy=1, să se arate că:

$$(1+x)(1+y)\geq 4$$

Dem.

$$1+x\geq 2\sqrt{x}$$

$$1+y\geq 2\sqrt{y}$$

$$-------------------$$

$$(1+x)(1+y)\geq 2\sqrt{x}\cdot 2\sqrt{y}$$

$$(1+x)(1+y)\geq 4\sqrt{xy}$$

$$(1+x)(1+y)\geq 4$$

Inegalitatea mediilor

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Ex4: Fie x,y numere reale pozitive astfel încât x+y=1, să se arate că:

$$x^2 + y^2 \geq \frac{1}{2}$$

Dem.

$$(x^2 + y^2)(1^2 + 1^2) \geq (x \cdot 1 + y \cdot 1)^2$$

Cauchy – Buniakovski - Schwarz

$$(x^2 + y^2) \cdot 2 \geq (x + y)^2$$

$$2(x^2 + y^2) \geq 1^2$$

$$x^2 + y^2 \geq \frac{1}{2}$$

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Ex5: Pentru orice număr real a demonstraţi inegalitatea:

$$a^4 - 2a^3 + 2a^2 - 2a + 1 \geq 0$$

$$a^4 - 2a^3 + 2a^2 - 2a + 1 \geq 0$$

$$a^4 + a^2 - 2a^3 - 2a + a^2 + 1 \geq 0$$

$$a^2(a^2 + 1) - 2a(a^2 + 1) + (a^2 + 1) \geq 0$$

$$(a^2 + 1)(a^2 - 2a + 1) \geq 0$$

$$(a^2 + 1)(a - 1)^2 \geq 0$$

$$(A)$$

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Ex6: Ordonaţi crescător numerele reale:

$$a^3; a^2; a; 1$$

I) $a \in (-\infty; -1)$

$a^3 < a < 1 < a^2$

II) $a = -1$

$a^3 = a < a^2 = 1$

III) $a \in (-1; 0)$

$a^3 < a < a^2 < 1$

IV) $a = 0$

$a^3 = a^2 = a < 1$

V) $a \in (0; 1)$

$a^3 < a^2 < a < 1$

VI) $a = 1$

$a^3 = a^2 = a = 1$

VII) $a \in (1; +\infty)$

$1 < a < a^2 < a^3$

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Ex7: Dacă a+b+c=0, demonstraţi că:

$$a^4+b^4+c^4=2(a^2b^2+a^2c^2+b^2c^2)$$

$$a+b+c=0$$
$$a+b=-c$$
$$a^2+2ab+b^2=c^2$$
$$a^2+b^2-c^2=-2ab$$
$$(a^2+b^2-c^2)^2=(-2ab)^2$$
$$a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2=4a^2b^2$$
$$a^4+b^4+c^4=2a^2b^2+2a^2c^2+2b^2c^2$$
$$a^4+b^4+c^4=2(a^2b^2+a^2c^2+b^2c^2)$$

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Ex8: Dacă a+b+c=0, demonstraţi că:

$$a^3+b^3+c^3=3abc$$

$$a+b+c=0$$

$$a+b=-c$$

$$(a+b)^3=(-c)^3$$

$$a^3+3ab(a+b)+b^3=-c^3$$

$$a^3+3ab(-c)+b^3+c^3=0$$

$$a^3-3abc+b^3+c^3=0$$

$$a^3+b^3+c^3=3abc$$

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Ex 9: Scrieţi sub forma de pătrat perfect expresia:

$$a(a+1)(a+2)(a+3)+1$$

$$a(a+1)(a+2)(a+3)+1=(a^2+a)(a^2+5a+6)+1=$$

$$=a^4+5a^3+6a^2+a^3+5a^2+6a+1=$$

$$=a^4+9a^2+1+6a^3+2a^2+6a=$$

$$=(a^2+3a+1)^2$$

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Ex 10: Scrieţi sub forma de pătrat perfect expresia:

$$(a-1)a(a+1)(a+2)+1$$

Fie a-1=t (folosim rezultatul de la problema precedentă)

$$t(t+1)(t+2)(t+3)+1=(t^2+3t+1)^2=[(a-1)^2+3(a-1)+1]^2=$$
$$=(a^2-2a+1+3a-3+1)^2=(a^2+a-1)^2$$

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Ex 11: Dacă $a^3+b^3+c^3=a^2+b^2+c^2=a+b+c=1$ atunci abc=0

$$a+b+c=1$$
$$a^2+b^2+c^2+2ab+2ac+2bc=1$$
$$2ab+2ac+2bc=0$$
$$ab+ac+bc=0$$

$$(a+b+c)(a^2+b^2+c^2)=1\cdot 1$$
$$a^3+ab^2+ac^2+a^2b+b^3+bc^2+a^2c+b^2c+c^3=1$$
$$ab(a+b)+ac(a+c)+bc(b+c)=0$$
$$ab(1-c)+ac(1-b)+bc(1-a)=0$$
$$ab-abc+ac-abc+bc-abc=0$$
$$-3abc+(ab+ac+bc)=0$$
$$-3abc=0$$
$$3abc=0$$

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Ex 12: Demonstraţi că: $a^{10} - a^{7} + a^{4} - a + 1 > 0$

$a \le 0$

$$\left.\begin{array}{l} a^{10} + a^{4} + 1 \ge 1 \\ -a^{7} - a \ge 0 \end{array}\right\} \Rightarrow a^{10} - a^{7} + a^{4} - a + 1 > 0$$

$0 < a < 1$

$$\left.\begin{array}{l} a^{10} > 0 \\ a^{4} - a^{7} > 0 \\ 1 - a > 0 \end{array}\right\} \Rightarrow a^{10} - a^{7} + a^{4} - a + 1 > 0$$

$a = 1$

$$a^{10} - a^{7} + a^{4} - a + 1 = 1 - 1 + 1 - 1 + 1 = 1 > 0$$

$0 < a < 1$

$$\left.\begin{array}{l} a^{10} - a^{7} > 0 \\ a^{4} - a > 0 \\ 1 > 0 \end{array}\right\} \Rightarrow a^{10} - a^{7} + a^{4} - a + 1 > 0$$

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Ex 13 Demonstraţi că pentru orice x>0 avem:

$$x+\frac{1}{2\sqrt{1+x^2}}<\sqrt{x^2+1}$$

Dem.

$$x+\frac{1}{2\sqrt{1+x^2}}<\sqrt{x^2+1}\Leftrightarrow\frac{1}{2\sqrt{1+x^2}}<\sqrt{x^2+1}-x\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{2\sqrt{1+x^2}}<(\sqrt{x^2+1}-x)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\Leftrightarrow\frac{1}{2\sqrt{1+x^2}}<\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{2\sqrt{1+x^2}}<\frac{1}{\sqrt{x^2+1}+x}\Leftrightarrow\sqrt{x^2+1}+x<2\sqrt{1+x^2}\Leftrightarrow x<\sqrt{1+x^2}\Leftrightarrow$$

$$\Leftrightarrow x^2<1+x^2(A)$$

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Ex 14

Demonstraţi că pentru orice numere reale a,b,c avem:

$$\frac{3(a^2+b^2+c^2)}{4} \geq \left(\frac{a+b+c}{2}\right)^2$$

Dem.

$$\frac{3(a^2+b^2+c^2)}{4} \geq \left(\frac{a+b+c}{2}\right)^2$$

$$\frac{3a^2+3b^2+3c^2}{4} \geq \frac{a^2+b^2+c^2+2ab+2ac+2bc}{4}$$

$$3a^2+3b^2+3c^2 \geq a^2+b^2+c^2+2ab+2ac+2bc$$

$$2a^2+2b^2+2c^2-2ab-2ac-2bc \geq 0$$

$$(a-b)^2+(b-c)^2+(c-a)^2 \geq 0 (A)$$

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Ex 15 Demonstraţi că pentru orice numere reale pozitive x,y,z avem:

$$x(y^2+z^2)+y(x^2+z^2)+z(x^2+y^2)\geq 2xy\sqrt{xy}+2yz\sqrt{yz}+2zx\sqrt{zx}$$

Dem.

$$xy^2+xz^2+yx^2+yz^2+zx^2+zy^2\geq 2xy\sqrt{xy}+2yz\sqrt{yz}+2zx\sqrt{zx}$$

$$xy^2-2xy\sqrt{xy}+yx^2+xz^2-2xz\sqrt{xz}+zx^2+yz^2-2yz\sqrt{yz}+y^2z\geq 0$$

$$xy(y+x-2\sqrt{xy})+xz(z-2\sqrt{xz}+x)+yz(z-2\sqrt{yz}+y)\geq 0$$

$$xy(\sqrt{x}-\sqrt{y})^2+xz(\sqrt{z}-\sqrt{x})^2+yz(\sqrt{y}-\sqrt{z})^2\geq 0(A)$$

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Ex 16 Demonstraţi că pentru orice numere reale pozitive a,b,c avem:

$$a^3+b^3+c^3 \geq ab\sqrt{ab}+ca\sqrt{ca}+bc\sqrt{bc}$$

Dem.

$$(a\sqrt{a}-b\sqrt{b})^2 \geq 0 \qquad (c\sqrt{c}-a\sqrt{a})^2 \geq 0 \qquad (b\sqrt{b}-c\sqrt{c})^2 \geq 0$$

$$a^3-2ab\sqrt{ab}+b^3 \geq 0 \qquad c^3-2ca\sqrt{ca}+c^3 \geq 0 \qquad b^3-2bc\sqrt{bc}+c^3 \geq 0$$

$$2a^3+2b^3+2c^3-2ab\sqrt{ab}-2ca\sqrt{ca}-2bc\sqrt{bc} \geq 0$$

$$a^3+b^3+c^3-ab\sqrt{ab}-ca\sqrt{ca}-bc\sqrt{bc} \geq 0$$

$$a^3+b^3+c^3 \geq ab\sqrt{ab}+ca\sqrt{ca}+bc\sqrt{bc}$$

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Ex 16 Demonstraţi că pentru orice numere reale a,b, dacă a+b=2 atunci avem:

$$a^2+b^2\geq 2$$

Dem.

$$a^2+b^2\geq 2$$

$$(2-b)^2+b^2\geq 2$$

$$4-4b+b^2+b^2\geq 2$$

$$2b^2-4b+4-2\geq 0$$

$$2b^2-4b+2\geq 0$$

$$b^2-2b+1\geq 0$$

$$(b-1)^2\geq 0(A)$$

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Ex 16

Demonstraţi că pentru orice numere reale a,b, dacă a+b=2 atunci avem:

$$a^3 + b^3 \geq 2$$

Dem.

$$a^3 + b^3 \geq 2$$

$$(a+b)(a^2 - ab + b^2) \geq 2$$

$$2(a^2 - ab + b^2) \geq 2$$

$$a^2 - ab + b^2 \geq 1$$

$$(2-b)^2 - (2-b)b + b^2 \geq 1$$

$$4 - 4b + b^2 - 2b + b^2 + b^2 - 1 \geq 0$$

$$3b^2 - 6b + 3 \geq 0$$

$$b^2 - 2b + 1 \geq 0$$

$$(b-1)^2 \geq 0 (A)$$

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Ex 17

Demonstraţi că pentru orice numere reale a,b,c,x,y,z dacă:

$$\begin{array}{l} a^2+b^2+c^2=1 \\ x^2+y^2+z^2=1 \end{array} \Rightarrow |ax+by+cz|\le 1$$

Dem.

Aplicăm teorema lui Cauchy – Buniakovski – Schwarz numerelor: a,b,c,x,y,z:

$$(a^2+b^2+c^2)(x^2+y^2+z^2)\ge(ax+by+cz)^2$$

$$1\cdot 1\ge(ax+by+cz)^2$$

$$(ax+by+cz)^2\le 1$$

$$-1\le ax+by+cz\le 1$$

$$|ax+by+cz|\le 1$$

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Ex 18 Demonstraţi că pentru orice numere reale pozitive a,b,c dacă:

$$a+b+c=1 \quad \Rightarrow \sqrt{a^2b+b^2c+c^2a} \geq ab+bc+ca$$

Dem. Aplicăm teorema lui Cauchy – Buniakovski – Schwarz numerelor:

$$a\sqrt{b};b\sqrt{c};c\sqrt{a}$$

$$\sqrt{b};\sqrt{c};\sqrt{a}$$

$$(a^2b+b^2c+c^2a)(b+c+a) \geq (ab+bc+ca)^2$$

$$(a^2b+b^2c+c^2a)\cdot 1 \geq (ab+bc+ca)^2$$

$$\sqrt{a^2b+b^2c+c^2a} \geq ab+bc+ca$$

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Ex 19 Demonstraţi că pentru orice numere reale pozitive x,y,z avem:

$$\frac{x}{-x+y+z}+\frac{y}{x-y+z}+\frac{z}{x+y-z}\geq 3$$

Dem. Fie: $x=a+b$

$y=b+c$

$z=a+c$

atunci $$\frac{x}{-x+y+z}+\frac{y}{x-y+z}+\frac{z}{x+y-z}\geq 3$$

$$\frac{a+b}{2c}+\frac{a+c}{2b}+\frac{b+c}{2a}\geq 3$$

$$\frac{a+b}{c}+\frac{a+c}{b}+\frac{b+c}{a}\geq 6$$

$$\frac{a}{c}+\frac{b}{c}+\frac{a}{b}+\frac{c}{b}+\frac{b}{a}+\frac{c}{a}\geq 6$$

$$(\frac{a}{c}+\frac{c}{a})+(\frac{b}{c}+\frac{c}{b})+(\frac{a}{b}+\frac{b}{a})\geq 6(A)$$

$$\frac{a}{c}+\frac{c}{a}\geq 2 \Leftrightarrow a^2+c^2\geq 2ac$$

$$\frac{b}{c}+\frac{c}{b}\geq 2 \Leftrightarrow b^2+c^2\geq 2bc$$

$$\frac{a}{b}+\frac{b}{a}\geq 2 \Leftrightarrow a^2+b^2\geq 2ab$$