1. SURVEY OF NUMBERS

The concept of numbers was the result of a very long process. It started at a time from which we have no written records. Even in early times, people wanted to enumerate different things. The need to enumerate created the numbers one, two, three, ..., which we call *positive integers.*

In a survey of the concept of numbers, we could follow the history of this subject, but in mathematical classes this is not the most effective way. We want to get to know the up-to-date concept of numbers and there is a straightforward way to do this.

We extend the set of positive integers with zero. (If there are twenty people in a classroom and all of them go out, then we can say that the number of people remaining in the room is zero).

The numbers 0, 1, 2, 3, ..., are called natural numbers.

Between the natural numbers we can introduce the operations of addition and multiplication. We have already studied these, and we know that

$$8+7 = 15, \quad 9+0 = 9, \quad 2+0+4 = 6,$$
$$3 \cdot 4 = 12, \quad 7 \cdot 1 = 7, \quad 5 \cdot 0 = 0, \ldots$$

The results of adding and of multiplying natural numbers are also natural numbers.

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Sometimes we can substract one natural number from another one. For example: $9-2 = 7$. In order to get, a meaningful result from any subtraction, we must extend the family of natural numbers (among the natural numbers the subtraction $6-9$ does not have any meaning). We introduce *negative integers.* These are the numbers -1, -2, -3, ...

The numbers ..., $-3, -2, -1, 0, 1, 2, 3, \ldots$ are called integers.

Between the integers, we can introduce the operations of addition, subtraction and multiplication. These operations are familiar to us. We know that

$$5+(-3) = 2, \quad -7+(-3) = -10, \quad 5-(-4) = 9,$$
$$5\cdot(-4) = -20, \quad (-4)\cdot(-7) = 28, \quad 9\cdot(-2)\cdot(-3) = 54, \ldots .$$

On adding, subtracting and multiplying integers, we get integers.

Sometimes we can divide an integer by an other one. For example, 18:6 = 3. On the other hand, the division 3:4 can not be carried out among the integers, i.e. the result cannot be an integer. In order to guarantee that any division among the integers should have meaning, we must extend the family of numbers. Therefore, we introduce rational numbers. We have already defined when two ratios are considered to be equal to each other. For example $\frac{3}{4}$ and $\frac{6}{8}$ stand for the same number: $\frac{3}{4}=\frac{6}{8}$.

Numbers of the form $\frac{a}{b}$ where *a* and *b* are integers, are called rational numbers. ("Rational" is a word of Latin origin, which originally had the meaning of "proportional". It also has other meanings, such as "sensible, logical, reasonable".) Obviously, integers are rational numbers too.

We have defined some operations between rational numbers. According to these we know that

$$\frac{2}{3}+\frac{3}{4}=\frac{17}{12}, \quad \frac{2}{3}\cdot\frac{4}{5}=\frac{8}{15}, \quad \frac{3}{5}:\frac{7}{8}=\frac{24}{35} .$$

Rational numbers can be written in decimal form, for example $\frac{3}{4} = 0.75$; $\frac{1}{3} = 0.\dot{3}$; $\frac{1}{7} = 0.\dot{1}4285\dot{7}$. Decimal numbers can be **finite, or infinite recurring decimal numbers.** Finite decimal numbers can also be considered to be recurring ones (using 0 in the notation): $\frac{3}{4} = 0.75\dot{0}$. Integers can also be written in this way: $\frac{8}{4} = 2 = 2.\dot{0}$, or $\frac{3}{1} = 3.\dot{0}$.

Some of the rational numbers were written as recurring decimal numbers. Can all rational numbers be represented in this way? On the basis of some examples, we may expect a positive answer.

It can be proved that any rational number can be represented in recurring decimal number form.

When we work out the division $\frac{a}{b}$, if we have a remainder after each step then it must certainly be equal to one of the integers $1, 2, 3, \ldots, b-1$, which means that we can have at most $b-1$ different remainders. Therefore, sooner or later one of these remainders will recur, and from that point on the remainders will recur periodically. Thus, the digits in decimal form will also recur periodically. Once we get a remainder equal to 0, then all the subsequent remainders will be equal to 0, and therefore the corresponding digits of the decimal form will also be 0's.

The converse of this statement is also true: any recurring decimal number can be written as the ratio of two integers.

Of course, we should prove this statement too, but this exceeds our knowledge at present.

Remark: When we write rational numbers in the form of recurring decimal numbers, we may conjecture that two different decimal numbers may represent the same number. For example, $1.\dot{0}$ and $0.\dot{9}$ look different, but both represent 1. It can be proved that they are equal to each other. However, this does not influence our previous statement. The rational numbers are recurring decimal numbers, and the recurring decimal numbers are rational numbers.

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Do we need to extend the family of numbers further, or is this enough for our purposes?

We saw in primary school that the square drawn on one of the diagonals of a unit square has an area equal to 2 (Figure 5). This square can not have a side whose length is equal to a rational number, because it can not be an integer ($1^2 = 1, 2^2 = 4$), nor a rational number of form $\frac{a}{b}$ (where *a* and *b* do not have a common divisor and *b* is greater than 1), since otherwise we get that the product $\frac{a}{b}\cdot\frac{a}{b}$ is not an integer (we shall return to this point later).

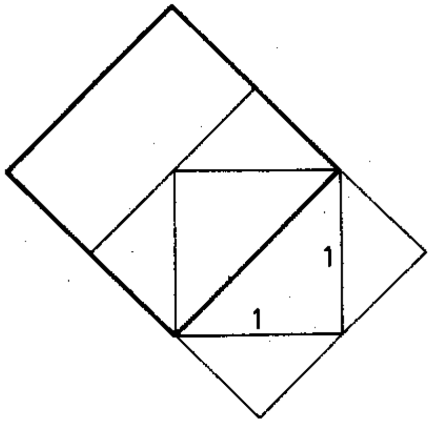

Figure 5

If we want to express the length of the side with a number, then we must extend the family of numbers once more. We must introduce a new type of number. We want to have a number whose square is equal to 2. In primary school, we introduced such a number and denoted it by $\sqrt{2}$. We accept that if we have two numbers a and b and $a < b$, than $a^2 < b^2$. In view of this, we get the following estimations:

$$\begin{aligned} 1 &< \sqrt{2} < 2, \\ 1.4 &< \sqrt{2} < 1.5, \\ 1.41 &< \sqrt{2} < 1.42, \\ 1.414 &< \sqrt{2} < 1.415, \\ &\vdots \end{aligned}$$

We can continue this approximation as long as we want. In this way we approximate $\sqrt{2}$ by decimal rational numbers: $\sqrt{2} = 1.4142 \ldots$. $\sqrt{2}$ is not rational, so in decimal form we can not find any recurring period. Such numbers are called *irrational numbers*. The word "irrational" is the opposite of the word rational, and means that the number can not be equal to the ration of two integers.

One example of irrational numbers is $\sqrt{2} = 1.414213\ldots$;

$\pi = 3.1415926\ldots$ too is an irrational number, but we can also construct irrational numbers by any rule which guarantees that there will be no recurring period in the sequence of digits. For example, the following number is irrational: 0.1010010001..., where there is a steadily increasing number of 0's between the 1's.

We need to define operations between irrational numbers too. In primary school we mostly worked with rational numbers. We knew the operations between them. We shall deal with the operations between irrational numbers later.

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We have established that rational numbers are recurring decimal numbers (and include the integers). We have introduced irrational numbers, which turned out to be non-recurring decimal numbers.

Infinite decimal numbers can be recurring or non-recurring, or in other words rational or irrational numbers. This leads us to give a particular name to **infinite decimal numbers.** We call them **real numbers.**

We extended the family of numbers in the following sequence natural numbers, integers, rational numbers, real numbers.

We spoke about numbers in primary school and represented them in the Venn diagram. The different families of numbers are represented in Figure 6.

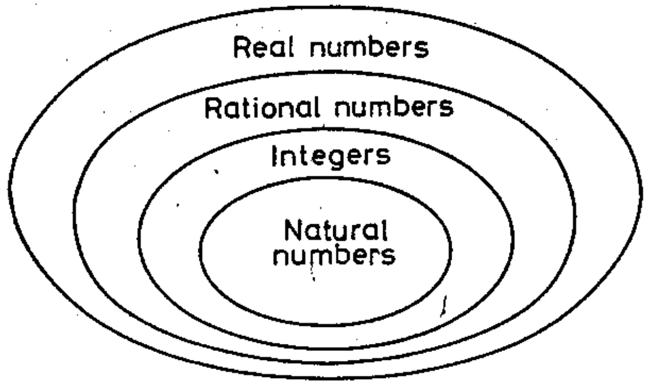

Figure 6

In order to be able to refer to the different families of numbers, we shall introduce some notations:

The set of natural numbers will be denoted by	**N.**
The set of integers will be denoted by	**Z.**
The set of rational numbers will be denoted by	**Q.**
The set of real numbers will be denoted by	**R.**

We use capital letters to denote sets. The above symbols, which refer to the Latin names, are used throughout the world.

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All numbers we have worked with so far have been real numbers. *For us, the term number meant real numbers,* for we did not know of other numbers. However, the question naturally arises: Is it possible to extend the concept of numbers further and introduce other numbers. The answer is yes, but we shall not go into details now.

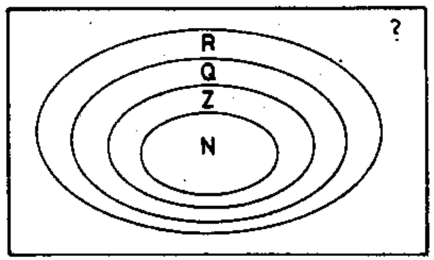

Figure 7

We could introduce *non-real num-*

bers and could also define operations on them. Therefore, instead of saying "numbers", we should really use the term "real numbers". We can extend the Venn diagram to non-real numbers too (Figure 7).

We have known the concept of the datum line (number line) for several years. After introducing real numbers, we can say that each point of the datum line corresponds to a real number, and each real number corresponds to a point of the datum line (Figure 8).

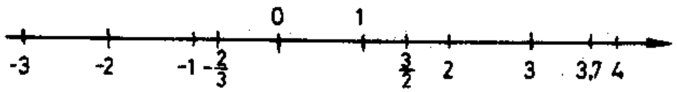

Figure 8

We find the above property to be obvious, but it is not self-evident at all. When determining the datum line, we choose two points. We denote these by 0 and 1. We connect these points by a straight line and, starting at the points 0 and 1, measure the distance between 0 and 1 on the line in both directions repeatedly. In this way we represent the numbers 2, 3, 4, ..., -1, -2, -3, -4, ... We can divide the distance between any two neighbouring points into two equal parts. We can then do the same with the new segments or, using a constructional method other than halving, we can divide the distance between two points into an arbitrary number of equal parts. In this way we can represent any rational number on the datum line. Between any two points there are infinitely many points, each corresponding to rational numbers.

On the other hand, none of these points corresponds to the length of the diagonal ($\sqrt{2}$) of a unit square, i.e. $\sqrt{2}$ is an irrational number. Thus, besides the infinitely many points corresponding to the rational numbers, there is space to represent irrational numbers.

We can construct a segment of length $\sqrt{2}$ and then, using a pair of compasses, we can mark off this segment on the datum line. Thus, at least "theoretically", we can denote the point corresponding to $\sqrt{2}$. On the other hand, using a ruler and a pair of compasses, we can not construct a segment of length $\pi = 3.1415926...$, which is a non-recurring decimal number. Therefore, the point corresponding to π can only be approximated.

We should have proved that the real numbers fill out the datum line completely, but here we omit the proof.

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We shall next state some basic concepts and statements about real numbers. Most of them are already known to you, but to continue the development it is important for you to learn these statements consciously.

a) From two real numbers, say -5 and 3.4, it easy to choose the larger one: $-5 < 3.4$.

It is useful to formulate precisely what "being larger" and "being smaller" mean. -5 is the smaller number since there exists a positive real number such that by adding it to the

smaller one, we get the larger number. This positive number is 8.4, for $-5 + 8.4 = 3.4$.

If we want to formulate the essence, it is superfluous to mention concrete numbers such as -5 and 3.4.

We have two real numbers a and b. We say that $a < b$ if there exists a positive real number d such that $b = a + d$. (This is called the *definition of ordering.*)

b) We define the **absolute value** (modulus) of a real number:

$$|a| = \begin{cases} a, & \text{if } 0 \leqq a, \\ -a, & \text{if } a < 0. \end{cases}$$

We get for example: $|a^2| = a^2$, $|-9| = 9$;

$$|x-5| = \begin{cases} x-5, & \text{if } 5 \leqq x, \\ -x+5, & \text{if } x < 5. \end{cases}$$

c) *The addition of real numbers is a commutative and associative operation.*
"Commutative" means: $a + b = b + a$.
When we add two real numbers together, we can change the order of the terms.
"Associative" means: $(a + b) + c = a + (b + c)$
When we add together more than two real numbers, we can group the terms together arbitrarily.

d) *Multiplication of real numbers is commutative and associative.*
"Commutative" means: $ab = ba$.
When we multiply together two real numbers, we can change the order of them.
"Associative" means: $(ab)c = a(bc)$.
When we multiply together more than two real numbers, we can group them together, arbitrarily.

e) *Multiplication is a distributive operation with respect to the product,* i.e.:

$$(a + b)c = ac + bc.$$

If we multiply a sum of real numbers by a real number, then we can get the same result by multiplying each term of the sum by the multiplier and then adding these products together. (We say that the product can be distributed between the terms.)

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We have mentioned that we survey the concept of numbers in a very straightforward way. Surveying the history of the concepts of numbers is not so easy. Many large groups of people created their own methods of enumeration and the order of operations they needed. The linguistic expressions and other written notes show that different number systems were created.

For us, the decimal system is a very natural one to use. This is a number system, of Hindu origin which was transmitted into Europe by the Arab around the 10th century. Until that time the Roman system was used, here, for example 1988 = MCMLXXXVIII. The Roman system is much harder to use than the decimal system and carrying out the operations is much slower and clumsier.

We do not know the exact time when the decimal system was introduced. The earliest written note (976) can be found in a Spanish cloister. The new system spread very slowly; many people were against it. The merchants were afraid that their bills would be easy to falsify because of the similar shapes of the numbers 6, 9 and 0. In Firenze, it was even prohibited to use this system. In the business notes of the famous Medici family, it was commonly used only after 1406. The first book with decimal page numbering was published only in 1471.

The oldest written document containing Hindu-Arabic numbers in Hungary dates from 1407. The numbers on the church in the community of Kalotaszeg Magyarvalkó (Rumania, Valeni) are mixtures of Arabic and Roman numbers. The date (1470) when the Hungarian national shield exhibited in the Mathias church was made can be seen in decimal form, with the use of Arabic numbers (Figure 9). During the reign of Ulászló II, in around 1499, coins were used which had the date of manufacture printed in Hindu-Arabic numbers. Coining in this way started in Switzerland in 1424, in Austria in 1484, in France in 1485, in England in 1551, and in Russia in 1654.

Picture 2

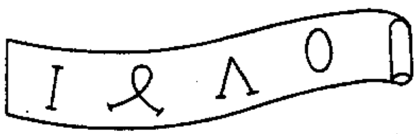

Figure 9

The shapes of the Hindu-Arabic numbers were slightly changed. We can find the recent shapes in *Dürer*'s (1471–1528) engraving entitled "Melancholy" (1514). It is assumed that Dürer designed the shapes of the recent Hindu-Arabic numbers.

Even in early times, the process of sharing required concepts such as "half" or "halving", but initially this had nothing to do with 1/2. Linguists have proved that in languages the word "half" can be related neither to 1 nor to 2. The concept of rational numbers was introduced much later. The Greeks described rational numbers around the 1st and 2nd centuries A. D. The Romans used only the concept of twelfth parts.

The idea of decimal numbers was introduced only when the decimal system had long been widely used. (Counting with the numbers of Hindu origin, and also with the sixtieths used by the Arabs, was more advanced compared with the Roman system.)

Picture 1

The first attempt to work out a system of decimal numbers was made around the 14th century. Decimal numbers can be found in books printed in the second half of the 16th century. The book by the Dutch mathematician *Stevin* (1548–1620) (originally a merchant, and later an engineer), published in 1585 and entitled "The tenth", was what not only convinced the public of the advantages of decimal numbers, but also suggested the use of tenths of the units of measurements in different calculations.

The widespread use of the decimal point was due to *Napier* (pronounced: népjor, 1550–1610) and that of the decimal comma to *Kepler* (1571–1630).

The negative numbers were for a long time considered to be "nonsense" numbers. The symbols + and − were first used by merchants referring to a surplus or a lack, and became mathematical symbols around the turn of the 15th and 16th centuries, but they became common only in the 17th century.

The existence of irrational numbers was already known in the 5th century B. C. The problem of determining the length of the diagonal of a unit square had led mathematicians to the idea of $\sqrt{2}$. They could approximate it, but even the Greeks were opposed to the concept of irrational numbers, although they got close to it. All they knew about irrational numbers was that they are not rational ones. At the end of the 16th century, it became known that rational numbers can be written in the form of recurring decimal numbers, while the decimal numbers corresponding to irrational numbers have no recurrence.

The number concepts which fit modern needs were introduced by *Dedekind* (1831–1916), and particularly *Cantor* (1845–1918), in the second half of the 19th century.

Exercises

1. Determine the values of the following expressions by mental arithmetic, i.e. without writing anything or using calculators. Write down only the results:

a) $\left(\frac{3}{4}+7.5-\frac{5}{2}+0.25\right)\cdot\left(-\frac{1}{2}+4.3+\frac{3}{2}+\frac{7}{10}\right)=$;

b) $(3.72+5+1.28)\cdot(4.3+1.2)=$;

c) $\frac{\frac{4}{1}}{2}+\frac{\frac{2}{1}}{6}+\frac{1}{0.1}=$;   *d)* $\frac{3.37+4.63}{\frac{3}{4}-\frac{2}{8}}=$.

2. Arrange the following numbers in increasing sequence and find the corresponding points on the datum line:

a) -3; $\frac{1}{2}$; -1.6; $\frac{9}{5}$; 0; $\frac{3}{2}$; $\frac{7}{5}$; $-\frac{3}{4}$; $-\frac{11}{5}$;

b) Add 2 to each negative number of *a)*, and subtract 2 from the positive numbers of *a)*, and then solve the problem again.

3. *a)* Arrange the following numbers in increasing sequence and find the corresponding points on the datum line:

$$\frac{0.37}{0.1};\quad \frac{6}{15};\quad -\frac{0.07}{0.1};\quad \frac{3}{5};\quad -\frac{13}{5};\quad \frac{12}{8};\quad -\frac{3}{4}.$$

b) Multiply each of the previous numbers by -1 and then arrange them in increasing sequence and find the corresponding points on the datum line.

4. Simplify the following expressions:

a) $\dfrac{3\cdot\left(0.7+\frac{3}{10}-\frac{1}{2}\right)+0.5}{0.7-\frac{9}{20}}$;   *b)* $\dfrac{2\cdot\left(4.6+\frac{2}{5}\right)-\frac{6}{2}}{3\cdot\left(1.3+\frac{17}{10}\right)+\frac{0.5}{0.1}}$

5. For what values of a will be the following absolute values be positive or equal to zero:

a) $|a-5|$;   *c)* $|2a+4|$;   *e)* $|2a-7|$;
b) $|a+3|$;   *d)* $|3a-6|$;   *f)* $|3a+5|$.

6. Write the following rational numbers in the form of recurring decimal numbers:

$$\frac{29}{37};\quad \frac{31}{303};\quad \frac{59}{74};\quad \frac{201}{185}.$$