

2. SETS AND SUBSETS

In Chapter 1 we mentioned certain sets of numbers.

The concept of a set is so simple and natural that we can not define it or reduce it to another one. *Set is considered to be a basic concept.*

We can circumscribe the concept of a set or illustrate it by some examples. A set can consist of different things, such as numbers, people, objects, points, or even concepts, etc. These are called the *elements of the sets*. (It should be mentioned that in mathematics the elements of sets are usually certain concepts, such as points, numbers, etc.)

After a while, everybody forms a picture of a set and finds certain properties which the set is expected to satisfy. We get a feeling, for example, that some of the elements of a set should themselves form a set, or the combination of two sets should also be regarded as a set. The number of elements belonging in a set can be finite; for example, the set of two-digit natural numbers has 90 elements. On the other hand, there are sets having infinitely many elements. One such set is the set of positive integers. Sets are denoted by capital letters. The elements will be listed between braces $\{ \}$.

Example 1. The set of one-digit odd integers: $A = \{1, 3, 5, 7, 9\}$.

Example 2. The set of two-digit positive numbers divisible by 29:
 $B = \{29, 58, 87\}$.

The symbol \in denotes that an element belongs in a set. In connection with Example 2: $9 \in A$ (read: "9 is an element of set A "), or $87 \in B$. If we want to denote that an element does not belong in a set, then we strike out the symbol \in , for example $8 \notin A$ (read: "8 is not an element of set A "), or $30 \notin B$.

We can speak about **empty sets** (if everybody leaves the classroom, then the set of people who are in the classroom is empty). *An empty set does not have an element. The symbol \emptyset stands for an empty set. The set $\{0\}$ is not an empty set, because it has one element, i.e. 0.*

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How can we define sets? You may have already observed that the *definition of a set is the same as the unique definition of its elements*.

a) If a set has finitely many elements, then we can define the set by *listing its elements*. For example, instead of speaking about the set of positive primes less than 20, we can refer to the set $C = \{2, 3, 5, 7, 11, 13, 17, 19\}$.

It is possible that a set of finitely many elements has so many elements that it is difficult to list them. Sets of infinitely many elements are not defined by listing the elements. We should look for another solution.

b) *When defining a set, we should give clear instructions to allow us to decide whether a certain element belongs in the set or not.*

It is very important that on the basis of these instructions we should *always* be able to determine whether an element belongs to the set or not. Otherwise, the instructions do not give a set. For example, we can not speak about the set of children with brown hair in a class if it is impossible to decide whether one particular child has brown or blonde hair.

The instructions determining a set will be enclosed by braces $\{ \}$. Immediately after the brace (we write a letter or an expression which stands for the elements of the set, then we draw a vertical line (or put a colon), and next we list the instructions which determine the elements of the set.

Example 3. Let D denote the set of one- and two-digit square numbers. Give another definition for D using standard mathematical notations.

Each element of set D has the form n^2 , where $1 \leq n < 10$ and n is a positive integer. The notation $n \in \mathbf{N}$ means that n is a positive integer. Therefore, the set in question is:

$$D = \{n^2 \mid 1 \leq n < 10 \text{ és } n \in \mathbf{N}\} \quad \text{or} \\ D = \{n^2 : 1 \leq n < 10 \text{ és } n \in \mathbf{N}\}.$$

Example 4. Describe the elements of the following set in words:

$$E = \{10a + 7 \mid 1 \leq a \leq 9 \text{ és } a \in \mathbf{Z}\}.$$

Since $1 \leq a \leq 9$ and $a \in \mathbb{Z}$, a can be one of the following nine numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9 (in view of the inequality $1 \leq a$, a must be positive and so the instruction $a \in \mathbb{N}$ instead of $a \in \mathbb{Z}$ would also have been correct). By multiplying the values of a by 10 and then adding 7, we get all the two-digit positive integers which end in 7.

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Examples 3 and 4 show that a set can be given in two or more different ways. For example, set E above is the same as set F :

$$F = \{\text{two-digit numbers ending in } 7\}.$$

We say that sets E and F are equal to each other. Here is the precise definition of the equality of two sets:

Two sets are considered to be the same if and only if their elements are the same. In other words, sets M and N are the same if and only if $a \in M$ implies $a \in N$, and $a \notin M$ implies $a \notin N$ too.

Example 5. Let S be the set of one-digit primes and one-digit even numbers.

$$S = \{2, 3, 4, 5, 6, 7, 8\}.$$

S has 7 elements. One of them is 2. The number 2 is both a prime and an even number. However, it does not have to occur twice in set S . We can not distinguish the prime 2 and the even number 2 from each other.

From now on, if we give a set by listing its elements, we list each element only once.

In Example 5, the elements were numbers. If the set consists of 25 students studying in the same class, then the set consists of 25 elements. Although each of them is a student, they are still different, so they are also different in the set.

We know that every rational number is a real number, and we have also seen that there are real numbers, such as $\sqrt{2}$, which are not rational numbers. We have the feeling that rational numbers form a subclass of real numbers. To ease our further work, we shall introduce the *concept of subsets*. We must give a definition which makes it possible to decide whether one set is a subset of another.

Definition: Set A is a subset of set H if each element of A belongs in H too. (In notation $A \subseteq H$. Read: " A is a subset of H ".)

The set of rational numbers is a subset of the real numbers. Briefly: $\mathbb{Q} \subseteq \mathbb{R}$, since sets \mathbb{Q} and \mathbb{R} satisfy the property that each element of \mathbb{Q} belongs in set \mathbb{R} too.

According to the definition of a subset, every set is a subset of itself.

An empty set is a subset of every set.

It also follows from the definition that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ (Figure 10.)

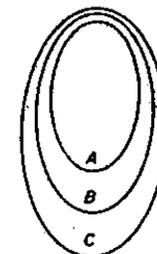


Figure 10

Example 6. List all the subsets of set $A = \{5, 7, 8\}$. It is easy to list some of the subsets, but it is useful to do it systematically.

An empty set is of course a subset. Further subsets may have one, two or three elements. Those subsets which have one element are $\{5\}$, $\{7\}$ and $\{8\}$. Consider the first element in the above definition of A . Now it is 5. Let 5 be the first element in a two-element subset. The second element can be 7 or 8 (which were listed in this sequence after 5 in the definition of set A). Next, let us take 7 (which is located next to 5 on going from left to right in the same list) as the first element of a two-element subset. The only choice for the second element providing a new subset is 8 (there is no other element to the right or 7 in the above list). Set A has only one three-element subset, namely itself. Therefore, the subsets of $A = \{5, 7, 8\}$ are:

$$\emptyset, \quad \{5\}, \quad \{5, 7\}, \quad \{5, 7, 8\}, \\ \{7\}, \quad \{5, 8\}, \\ \{8\}, \quad \{7, 8\}.$$

Of course, there are other ways to list the subsets. The only important thing is that we should work systematically, as otherwise it would be very difficult to list all the subsets of a set which consists of a large number of elements. We shall simply mention (without proof) that a 4-element set has $2^4 = 16$ subsets, a 5-element set has $2^5 = 32$ subsets, ..., and an n -element set has 2^n subsets.

Example 7. Find the relation between sets $G = \{2, 3, 5\}$. Both sets have the same elements, so they are equal to each other: $G = K$. From the definition of a subset, it follows that $G \subseteq K$ (each element of G belongs in K too). Similarly, $K \subseteq G$ also holds. On the other hand, the opposite statement is also true: from the properties $G \subseteq K$ and $K \subseteq G$, it follows that $G = K$.

In Example 7, the notation $G \subseteq K$ may seem unnatural if $G = K$. This is because of our ideas in everyday life. For example, if somebody asks for a piece of a cake then we may suppose that he will not eat all of the cake. This is why we introduce the *concept of true subsets*:

Definition: Set A is a true subset of set H if A is a subset of set H but is not equal to it. (In notation, $A \subset H$; read: "Set A is a true subset of set H ".) Briefly: $A \subset H$ if $A \subseteq H$ and $A \neq H$.

Returning to the example of rational and real numbers, we can say that $\mathbb{Q} \subset \mathbb{R}$. Earlier, we said that $\mathbb{Q} \subseteq \mathbb{R}$. Both are correct. The only difference is that statement $\mathbb{Q} \subset \mathbb{R}$ contains more information about the sets.

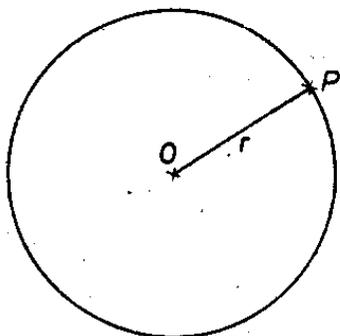


Figure 11

Denoting the given point by O and the given distance by r , the circle can be defined as the set of points P in the plane in the following way:

$$\text{Circle: } \{P \mid OP = r\}.$$

The perpendicular bisector of a line segment in a plane is the set of points which lie at the same distance from the end-points of the given line segment (Figure 12/a).

We earlier learned how to draw the perpendicular bisector of a line segment: Open a pair of compasses wide enough so that, when two circles are drawn around the end-points of the given line segment, the circles should intersect each other at two points, and then connect these two intersection points by a straight line (Figure 12/b).

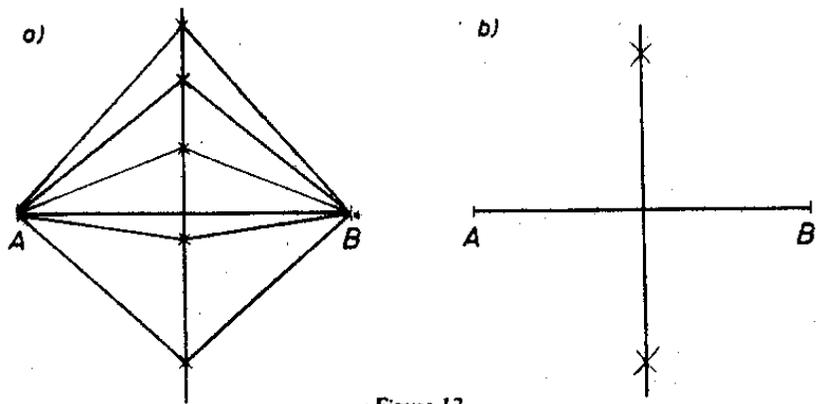


Figure 12

So far, most of our sets have consisted of numbers. We shall now give other examples from geometry. We are very familiar with the concept of a circle. A circle was defined as a set of points (Figure 11).

A circle is a set of points in a plane which lie at the same distance from a given point.

First we gave a definition for the perpendicular bisector of a line segment, and then we described a method of drawing it. How can we know that every point of the resulting line satisfies the definition of a perpendicular bisector? And how can we know that there are no more points satisfying the definition?

Rigorous thinking does not allow us to avoid these questions, but we shall mention with this until a little later.

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Since each real number on the datum line corresponds to a point, and each point corresponds to a real number, we can say that the set of real numbers can be represented by the points of a datum line. This makes it possible to represent different sets of numbers on a line, which is often very convenient.

Consider the set of real numbers given by the following inequalities $-2 \leq x \leq 3$. Using our notations:

$$A = \{x \mid -2 \leq x \leq 3, x \in \mathbb{R}\}.$$

Find the points corresponding to the numbers -2 and 3 on the datum line. These two points and the points belonging in the interior of the line segment determined by these two points represent set A . In Figure 13, we have denoted this set by a thicker line. The end-points are small dark circles, to make it clear that they also belong in set A .

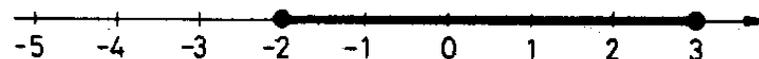


Figure 13

We can represent set B , given by the inequalities $-4 < x < 1$, in a similar way (Figure 14):

$$B = \{x \mid -4 < x < 1; x \in \mathbb{R}\}$$

However, this time the end-points are empty circles, meaning that they do not belong in set B .

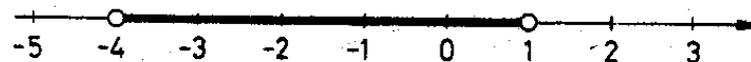


Figure 14

In order to make it easier to speak about such sets, we introduce the *concept* and the notation of *intervals*. The intervals in Figures 13 and 14 will be distinguished from each other. The former will be called a *closed in-*

terval, and the latter an *open interval*. In the usual notations, these line segments are: $-2; 3$ and $-4; 1$.

Definition: The closed interval $[a; b]$ is the set of real numbers for which $a \leq x \leq b$. The open interval $] a; b [$ is the set of real numbers for which $a < x < b$. For an open interval we also use the notation (a, b) .

We also speak about intervals closed (open) from the left and intervals open (closed) from the right. The appropriate notations:

The interval $-2 \leq x < 3$ is $[-2; 3[$, and
the interval $-5 < x \leq -1$ is $] -5; -1]$.

The word *interval* is also used for a line segment connecting two points on the datum line. In a similar way, these intervals can also be open, closed, open (closed) from the left, or closed (open) from the right.

Exercises

7. For each of the following integers, give the set of its digits:

- | | | |
|----------|-------------|---------------|
| a) 1988; | d) 2424; | g) 110; |
| b) 1989; | e) 240 440; | h) 1 000 000; |
| c) 1990; | f) 110 001; | i) 224 244. |

Are there equal sets among them?

8. Set A consists of the final digits of the square numbers. List the elements of A .

9. In a village are 100 men and a professional hair-dresser (who is also a man). Every man's beard is shaved off every morning. There are only two possibilities: a man shaves himself, or he goes to the hair-dresser to be shaved. Can we speak simply about the set of "people who shave themselves", and the set of "people who are shaved by the hair-dresser and not by themselves"?

10. Describe the following sets in words (see Example 4):

- $A = \{10k + 7 \mid k \in \mathbb{N}\}$;
- $B = \{n^2 \mid 3 < n < 10 \text{ and } n \in \mathbb{N}\}$;
- $C = \{n^2 + 12 \mid n \in \mathbb{N}\}$;
- $D = \{n^2 + 12 \mid 1 \leq n < 10 \text{ and } n \in \mathbb{N}\}$.

11. Write down the following definitions by using mathematical symbols:

- {positive integers in which the final digit is 0};
- {positive integers in which the final digit is 9};
- {integers which are twice a square number};
- {integers which are larger than a square number by 3}.

12. Let $A = \{10, 11, 12, 13, 14, 15\}$. Write down the elements of set $B = \{n^2 \mid n \in A\}$. Write down the subset of B consisting of elements which are:

- | | |
|--------------------|---------------------|
| a) even numbers; | d) divisible by 16; |
| b) divisible by 4; | e) divisible by 25; |
| c) divisible by 9; | f) primes. |

13. Among the following sets, find pairs where the first is a subset of the second:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$;
- $A = \{\text{numbers divisible by } 12\}$,
 $B = \{\text{numbers divisible by } 6\}$,
 $C = \{\text{even numbers}\}$,
 $D = \{\text{multiples of } 2\}$,
 $E = \{2\}$, $F = \{2, 4, 6\}$;
- $P = \{\text{quadrilaterals}\}$,
 $Q = \{\text{trapezia}\}$,
 $R = \{\text{parallelograms}\}$.

14. Write down the subsets of the sets:

- a) $\{3, 4\}$; b) $\{x, y, z\}$; c) $\{a, b, c, d\}$.

15. Write down the true subsets of the sets:

- a) $\{a, b\}$; b) $\{p, q, r\}$; c) $\{1, 2, 3, 4\}$.

16. List those subsets of set $A = \{1, 2, 3, 4, 5\}$ which contain the elements 3 and 5.

17. Find the following intervals on the datum line:

- | | | |
|---|-------------------------------------|---|
| a) $[-3; 0[$; | c) $\left[1; \frac{19}{2}\right]$; | e) $\left[-\frac{28}{8}; -1.5\right]$; |
| b) $\left]-\frac{28}{8}; -\frac{1}{2}\right[$; | d) $]2.5; 5.5]$; | f) $\left]-2\frac{1}{2}; 0\right[$ |

18. Use the introduced notations to write down the intervals which consist of real numbers

- larger than 2 and smaller than 7;
- not smaller than 2 and smaller than 7;
- larger than 2 and not larger than 7;
- not smaller than 2 and not larger than 7.

19. Among the following intervals, find pairs where the first is a subset of the second:

$$\begin{array}{lll}
 a)]-3; 4[; & c)]0; 0.5]; & e) \left] \frac{5}{2}; 4[; \\
 b) \left[-\frac{6}{3}; 3.5 \right]; & d) \left[-1.5; \frac{3}{2} \right[; & f) \left] -2; \frac{7}{2} \right].
 \end{array}$$

20. Among the following intervals, there are some which are subintervals of other intervals. We say that they form a descending chain if they can be expressed in the following way: $\dots \subset \dots \subset \dots \subset \dots$. Find such chains:

$$[-1; 4]; [0; 3]; [1; 2]; [-0.5; 4]; \left[\frac{3}{2}; 2.5 \right]; [0.1; 3].$$

3. SET-OPERATIONS

Exercise: Classify the numbers $10 \leq n < 30$ ($n \in \mathbf{N}$) with respect to divisibility by 3 and by 4.

Obviously, the above twenty numbers include some which can be divided by 3, some which can be divided by 4, and some which can be divided both by 3 and by 4. We can illustrate the classification in the Venn diagram. In order for the diagram to correspond with the written material, we introduce some notations. Let

$$\begin{aligned}
 H &= \{n \mid 10 \leq n < 30, n \in \mathbf{N}\}, \\
 A &= \{3k \mid 4 \leq k < 10, k \in \mathbf{N}\}, \\
 B &= \{4l \mid 3 \leq l \leq 7, l \in \mathbf{N}\}.
 \end{aligned}$$

These sets are illustrated in Figure 15. Our aim is to describe the sets consisting of those elements of H which

- can be divided by either 3 or 4,
- can be divided both by 3 and by 4,
- can be divided by 3 but not by 4, or can be divided by 4, but not by 3,
- can be divided by only one of the numbers 3 and 4,
- can not be divided by 3.

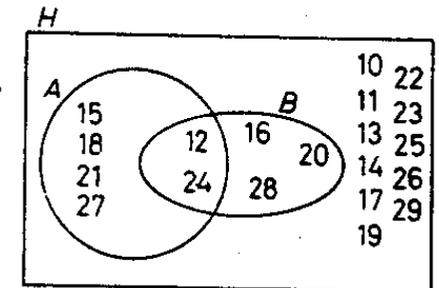


Figure 15

In cases a)–e) we look for new sets consisting of certain elements of sets H , A and B . These sets must satisfy different conditions. We shall determine these sets by using certain *set operations*, which we shall first define:

a) Union of sets

Let the above two sets A and B be given. We want to find the set whose elements can be divided by either 3 or 4.

In accordance with the meaning of the conjunction "either ... or ...", we look for numbers which can be divided by at least one of the numbers 3 and 4. These numbers belong in at least one of sets A and B .

Figure 16 illustrates sets A and B . The set we are speaking about here indicated by the shading lines. It can be seen that we have unified sets A and B , or in other words we have formed the "union of sets A and B ". We must give a precise definition of this operation:

Definition: The union of sets A and B is the set of all objects that are elements either of A or of B , or of both. The union is denoted by $A \cup B$.

Here, and everywhere else in mathematics, when we say "either ... or ...", we allow the possibility that both of the stated conditions hold. If we really mean "... but not both", we have to say so. We can define the union of n sets. The union of n sets $(A_1 \cup A_2 \cup \dots \cup A_n)$ is the set of objects which belong in at least one of the n sets.

Figure 17 illustrate the union of three sets (this union is indicated by the shading lines).

The following properties follow from the definition:

1. $A \cup B = B \cup A$ (commutativity);
2. $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$ (associativity);
3. $A \cup \emptyset = A$;
4. $A \cup A = A$.

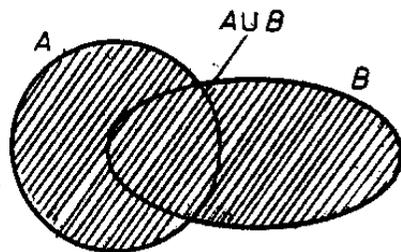


Figure 16

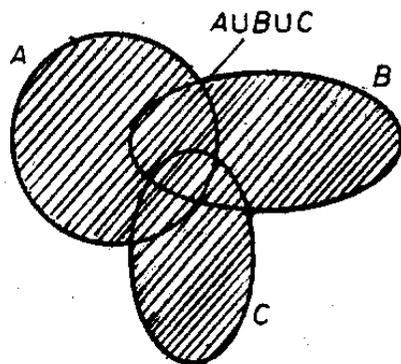


Figure 17

We can illustrate the above properties of the union operation in the Venn diagram, but this does not mean that we get a proof in this way. A proof should be based on the definition. Here, we show how to prove the second property.

According to the definition of the union operation, $A \cup B$ is the set of elements which belong in at least one of sets A and B . Set $(A \cup B) \cup C$ by definition consists of elements which belong in at least one of sets $A \cup B$ and C , i.e. in at least one of sets A , B and C . By definition, elements which are in at least one of sets B and C belong in set $B \cup C$. Also by definition, set $A \cup (B \cup C)$ consists of elements which belong in at least one of sets A and $B \cup C$, i.e. in at least one of sets A , B and C . We have got the same set, and therefore $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$.

b) Intersection of sets

Again, sets A and B be given let. We want to find the set whose elements can be divided both by 3 and by 4.

In accordance with the meaning of the conjunction "both ... and ...", we look for those numbers which can be divided by 3 and also by 4. These numbers belong both in set A and in set B .

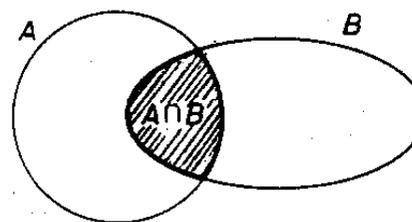


Figure 18

Figure 18 illustrates sets A and B . The set we are speaking about is indicated by the shading lines. We say it is the common part or "the intersection" of the two sets, or sometimes that it is "the product" of the two sets.

Definition: The intersection of sets A and B is the set of all objects that are elements of A and also elements of B . The intersection is denoted by $A \cap B$ (this is pronounced " A cap B ", because the symbol \cap looks like a cap, or " A intersection B ").

We can define the intersection of n sets. The intersection of sets A ($i = 1, 2, \dots, n$), i.e. sets $A_1 \cap A_2 \cap \dots \cap A_n$, is the set of those objects which belong in all of the n sets.

Figure 19 shows the intersection of three sets. The intersection we are speaking about is indicated by the shading lines.

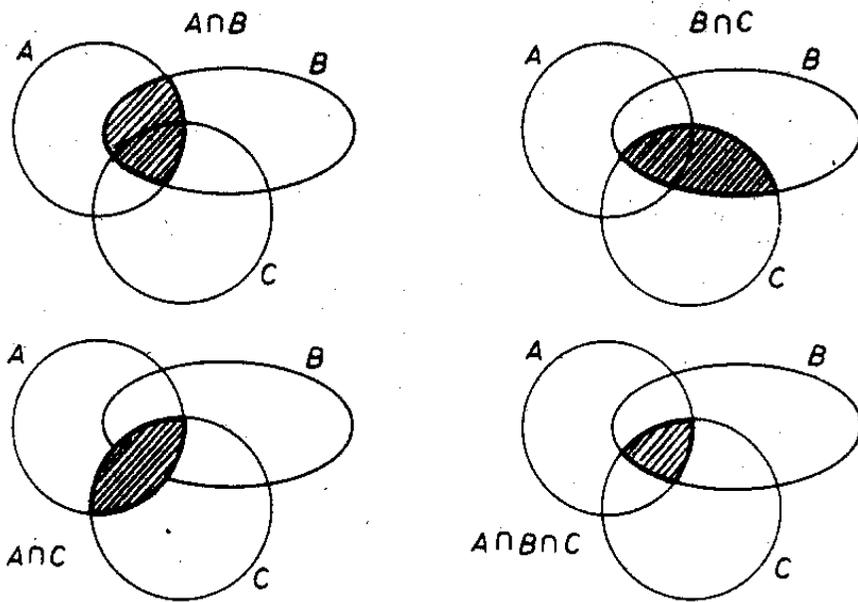


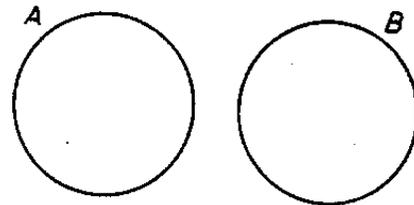
Figure 19

The following properties follow from the definition:

1. $A \cap B = B \cap A$ (commutativity);
2. $A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$ (associativity);
3. $A \cap \emptyset = \emptyset$;
4. $A \cap A = A$.

Remarks: 1. A word of caution is in order about the use of the word intersection. When we speak about the intersection of A and B , and write $A \cap B$, this allows the possibility that $A \cap B$ is \emptyset (Figure 20). However, when we say that two sets A and B intersect, we always mean that A and B have at least one element in common. This distinction in usage, between the noun and the verb, is not logical, but it is convenient; further, it is nearly universal, and there is not much to be done about it.

2. If sets A and B do not have a common element, i.e. $A \cap B = \emptyset$, then we also say that sets A and B are disjoint.



$$A \cap B = \emptyset$$

Figure 20

c) Difference of sets

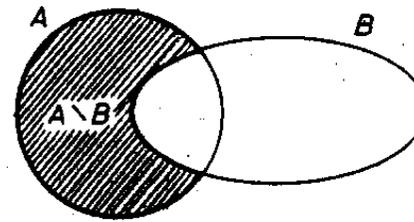


Figure 21

Consider sets A and B as before. We want to find the set whose elements can be divided by 3, but can not be divided by 4.

We now need to find the elements of set A which do not belong in set B ; we have to take away from set A those elements which belong in set

B . The derived set is indicated by the shading lines in Figure 21. This set is called the *difference of sets A and B*.

Definition: The difference between two sets A and B (in this sequence) is the set of all objects that belong in A but not in B . The difference is denoted by $A \setminus B$ (this is pronounced " $A - B$ ").

The following properties follow from the definition:

1. $A \setminus A = \emptyset$,
2. $A \setminus \emptyset = A$,
3. $\emptyset \setminus A = \emptyset$.

We have used the letters N , Z , Q and R to denote some well-known number sets. Using the above set operations, we can say the following:

The set of positive integers is $N \setminus \{0\}$. We denote it by N^+ :

$$N^+ = N \setminus \{0\} = \{\text{positive integers}\}.$$

Similarly:

$$Z^- = Z \setminus N = \{\text{negative integers}\},$$

$$R \setminus Q = \{\text{irrational numbers}\},$$

$$Z^- \cup N = Z, \text{ etc.}$$

d) Symmetrical difference of sets

Sets A and B are again given. Find the sets of those elements which can be divided by 3 or else by 4.

The conjunction "or else" (involving the excluding or") permits only one of the properties to hold in contrast with the conjunction "either ... or ..." (concessive or), which allows three possibilities: the first holds, the second holds, or both of them hold.

Now, in view of the "or else" the numbers divisible both by 3 and by 4 are excluded. The derived set is shown in Figure 22, indicated by the shading lines.

Definition: The symmetrical difference of sets A and B is by definition set $(A \setminus B) \cup (B \setminus A)$. By notation: $A \Delta B$ (this is pronounced "A delta B").

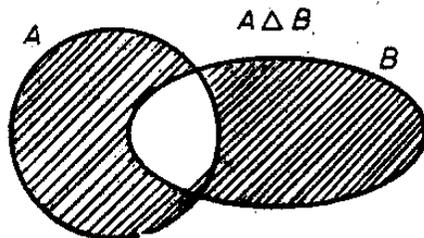


Figure 22

In other words, $x \in A \Delta B$ means that x is an element of exactly one of sets A and B , i.e.

$$A \Delta B = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}.$$

The following properties follow from the definition:

1. $A \Delta B = B \Delta A$ (commutativity);
2. $A \Delta B \Delta C = (A \Delta B) \Delta C \doteq A \Delta (B \Delta C)$ (associativity);
3. $A \Delta \emptyset = A$;
4. $A \Delta A = \emptyset$.

e) Complement of sets

What are the numbers in H which can not be divided by 3? All of the previous problems were about the elements of H . H was the fundamental set. In the operations union, intersection, difference and symmetrical difference, only sets A and B (subsets of H) were involved, but now we must take away elements of H not belonging in A . We introduce a new term for the remaining set.

Definition: $H \setminus A$ will be called the complement of set A . In notation: \overline{A} (this is pronounced "A bar"). This situation is depicted in Figure 23.

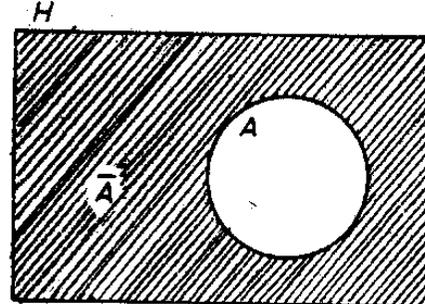
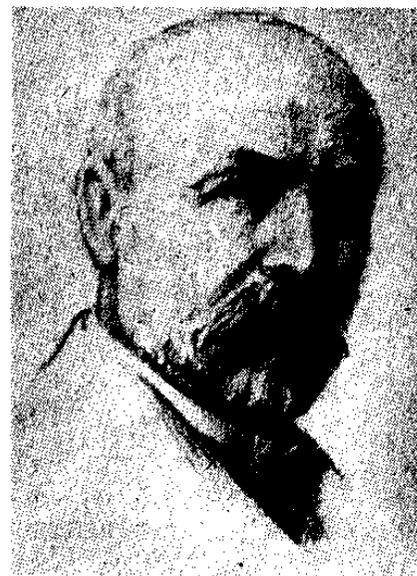


Figure 23



Picture 3

The following properties follow from the definition:

1. $\overline{\overline{A}} = A$;
2. $\overline{A \cap B} = \overline{A} \cup \overline{B}$;
3. $\overline{\overline{A}} = A$;
4. $\overline{H} = \emptyset$;
5. $\overline{\emptyset} = H$.

Around 1870, *G. Cantor* (1845–1918) created a new branch of mathematics, *set theory*. He made sets independent of their properties and considered them in a very abstract way. He realized that one can introduce concepts such as "smaller" and "larger" between sets of infinitely many elements. Set theory is still a developing part of mathematics, and its results can often be applied in other areas of mathematics. Here we have considered only the very basic concepts which will help us to formulate many facts and thoughts. With some exaggeration, we could say that what we have learned is mathematical shorthand writing.

Exercises

21. Two sets are given:

a) $A = \{4, 7, 8\}$ and $B = \{3, 4, 8, 9\}$;

b) $A = \{a, b, x, y\}$ and $B = \{b, c, x, y, z\}$.

Write down the elements of sets $A \cup B$, $A \cap B$, $A \setminus B$, and $B \setminus A$. Represent them in Venn diagrams.

22. Three sets are given:

a) $A = \{2, 3, 5, 8\}$, $B = \{1, 3, 4, 8\}$ and $C = \{1, 5, 6, 7\}$;

b) $A = \{a, d, e, f\}$, $B = \{a, c, d, g\}$ and $C = \{b, c, d, f, g\}$;

c) $A = \{a, b, c, c, x\}$, $B = \{b, c, z\}$ and $C = \{b, e, x, y\}$.

Write down the elements of sets $A \cup B$, $A \cap B$, $B \cup C$, $A \cap C$, $A \cup B \cup C$, $A \cap B \cap C$, $(A \cup B) \setminus C$, $(AB) \cap C$, and $B \setminus (A \cup C)$. Represent them in Venn diagrams.

23. Express the following intervals in a short way:

- a) $[1; 3] \cup [2; 7]$; b) $[-2; 3] \cap [-1; 7]$;
 c) $]3; 4] \cup [4; 5[$; d) $[-3; 2] \setminus [1; 3]$.

24. Three sets are given:

$$A = \{1, 2, 6, 7\}, B = \{1, 3, 5, 7\} \text{ and } C = \{2, 3, 4, 7\}.$$

List the elements of the following sets and sketch the corresponding Venn diagrams:

- a) $(A \cap B) \cup C$,
 b) $(A \cup C) \cap (B \cup C)$,
 c) $(A \cup B) \cap C$,
 d) $(A \cap C) \cup (B \cap C)$.

25. On the basis of the Venn diagrams corresponding to Exercises 24/a and 24/b, we may conjecture that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ holds. Prove this for arbitrary sets A , B and C (you can not argue by using the Venn diagram).

26. Similarly as in Exercise 25, conclude and prove identities motivated by the Venn diagrams corresponding to Exercises 24/c and 24/d.

27. Consider the set

$$H = \left\{ -0.81, -\frac{11}{5}, 1, -\frac{4}{2}, -\frac{5}{7}, -0.5, \frac{10}{9}, -11, -\frac{2}{5} \right\}.$$

List the elements of the following subsets and represent them in the corresponding Venn diagrams:

- a) $A_1 = \{\text{elements of } H \text{ which are smaller than } -2\}$,
 $A_2 = \{\text{elements of } H \text{ which are larger than } 1\}$,
 b) $B_1 = \{\text{elements of } H \text{ which are larger than } -2\}$,
 $B_2 = \{\text{elements of } H \text{ which are smaller than } 1\}$,
 c) $C_1 = \{\text{elements of } H \text{ which are smaller than } -1/2\}$,
 $C_2 = \{\text{elements of } H \text{ which are larger than } 1/2\}$,
 d) $D_1 = \{\text{elements of } H \text{ which are smaller than } -1/2\}$,
 $D_2 = \{\text{elements of } H \text{ which are not smaller than } -1/2\}$,
 e) $E_1 = \{\text{elements of } H \text{ which are not larger than } -1/2\}$,
 $E_2 = \{\text{elements of } H \text{ which are not smaller than } -1/2\}$.

28. Determine how many students from a class of 35 are members of the three available study circles (the history, the literature and the mathematics circles), of we know only the following information:

Eight students are each members of two study circles, 4 of these 8 are

members of the mathematics study circle, and there is only one who is a member of both the mathematics and the literature study circles. The number of students who are members of only the history study circle is equal to the number of students who are members of both the history and the mathematics study circles. Among the students who are members of the mathematics study circle, there are exactly as many who are members of only one circle as the number who are members of exactly two study circles. Five students are members only of the literature study circle, and there are 9 lazy students who are not members of any of the three study circles.

29. Some classmates had the following conversation:

- In our class everybody takes part in some physical training.
- Three of us take part in all three sports.
- Exactly half of us cycle.
- And half of them take part only in that sport.
- Five of those who cycle take part in two different sports.
- Two of the tennis players also cycle, but they do not swim.
- The number of children who only cycle is equal to the number of those who take part in exactly two sports.
- The number of children who play only tennis is less than the number of children who only swim by exactly one.

How many children cycle, how many play tennis and how many swim?